Abductive Reasoning with Type 2 Fuzzy Sets

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Abstract - **In fuzzy abduction, one needs to evaluate the membership distribution of the premise (antecedent clause), when the membership distribution of the consequent clause, and the fuzzy implication relations between the antecedent and the consequent clauses are provided. The paper formulates and solves the problem of fuzzy abduction by using type-2 fuzzy sets. It presumes background knowledge about the primary and the secondary antecedent to consequent implication relations to uniquely determine the type-2 fuzzy set corresponding to the antecedent clause, when the same for the consequent clause is provided. The proposed methodology of abduction would serve many interesting applications on predictions, forecasting, and diagnosis, where the environmental factor can be modeled with type-2 secondary distributions.**

Index Terms: Type-2 fuzzy sets, abductive reasoning, fuzzy modus ponens.

I. INTRODUCTION

The logic of fuzzy sets, introduced by Zadeh in 1965 [1], has undergone a dramatic evolution over the last four decades. Type-2 fuzzy sets, which too was proposed by Zadeh in 1975 [2], and was recently extended and popularized by Mendel [3-8], outperforms its type-1 counterpart in modeling real world uncertainty [9-10]. Type-2 fuzzy logic is currently being applied in various real world problems, including liquid-level control [11], medical diagnosis [12], and path-planning of robots in outdoor environment [13]. In this paper, we present a solution to the well-known problem on *fuzzy abduction* using type-2 fuzzy sets.

In the treatise of artificial intelligence (AI) [14], the notion of inferring the *fact p* from the rule: *If p then q,* and *the fact q* is called *abduction*. It is indeed important to note that abduction, though defined in classical AI, is not tractable in the logic of propositions or predicates. However, because of its inherent capability of approximate matching [15], the logic of fuzzy sets does support abduction. For example, let us consider a fuzzy rule: *If x is A, then y is B*, and a fuzzy proposition *y is B′*, where *x* and *y* are fuzzy linguistic variables, and *A, B, and B′* are fuzzy sets, such that $B' \approx B$. Then, following the principle of abductive reasoning, we can infer *x is A′ where* $A' \approx A$.

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There exists several works on fuzzy abduction [16-17] using type-1 fuzzy sets. However, to the best of our knowledge, no work has been reported on type-2 fuzzy abduction. The current paper is devoted to the discrete formulation of the type-2 abductive reasoning problem using fuzzy relations. It presumes that users should have background knowledge about primary and secondary antecedent to consequent implication relations. Given the primary and secondary membership profile of a consequent clause, we can easily evaluate the membership profile of the antecedent clause using inverse fuzzy relations. For the sake of simplicity and page length restriction, we presume transpose of a matrix to be its inverse with respect to max-min composition operation. However, a better inverse relation [18-20] can definitely improve the quality of the solution.

The rest of the paper is organized in the following manner. Section II provides some important definitions on type-1 and -2 fuzzy sets and relations. In section III, we outline generalized fuzzy modus ponens for type-1 and type-2 systems. Section IV presents an algorithm for fuzzy abduction. Examples are given in section V to illustrate the algorithm for abductive reasoning. A performance evaluation metric is defined in section VI to examine the intuitive justification of abduction through the examples worked out in section V. The conclusions arrived at the end of the paper and directions of future research are summarized in section VII.

II. MATHEMATICAL FOUNDATIONS

In this section, we define a few terminologies related to Type-1 (T1) and Type-2 (T2) fuzzy sets. These definitions will be used throughout the paper. Let X and Y be two universes of discourses and $A \subseteq X$ and $B \subseteq Y$ be two fuzzy sets.

Definition 1: A Type-1 fuzzy set A is characterized by a T1 membership function $\mu_A(x)$, where $x \in X$. A is defined by:

$$A=\{x,\mu_A(x)\} \tag{1}$$

where $0 \le \mu_A(x) \le 1$. Alternatively, A can be expressed as:

$$A = \int_{x\in X} \mu_A(x) \mid x \tag{2}$$

where $\int$ denotes union over all admissible x. For discrete universe of discourse, $\int$ is replaced by $\sum$.

Definition 2: A Type-2 fuzzy set $\tilde{A}$ is characterized by a T2 membership function $\mu_{\tilde{A}}(x,u)$, where $x \in X$ and $u \in J_x \subseteq [0, 1]$. $\tilde{A}$ is defined by:

$$\tilde{A} = \{((x,u),\mu_{\tilde{A}}(x,u)) \mid \forall x \in X, \forall u \in J_x \subseteq [0,1]\}. \tag{3}$$

Alternatively, $\tilde{A}$ can be expressed as:

$$\tilde{A} = \int_{x\in X}\int_{u\in J_x} \mu_{\tilde{A}}(x,u) \mid (x,u), J_x \subseteq [0,1] \tag{4}$$

where ∬ denotes union over all admissible x and u. For discrete universe of discourse, ∬ can be replaced by ∑∑.

Definition 3: A Type-1 fuzzy relation ${}^1R(x, y)$ between two T1 fuzzy sets A and B, where $x \in A \subseteq X$ and $y \in B \subseteq Y$ is defined as a two tuple and is given by:

$$^1R(x,y) = \{(x,y), \mu_R(x,y) \mid \forall (x,y) \in (X \times Y)\} \quad (5)$$

where $0 \le \mu_R(x,y) \le 1$.

Definition 4: A Type-2 fuzzy relation ${}^2R(x, u, y, v)$ between two T2 fuzzy sets $\widetilde{A} = \{(x,u), \mu_{\widetilde{A}}(x,y) \mid \forall x, u \in J_x \subseteq [0,1]\}$ and $\widetilde{B} = \{(y,v), \mu_{\widetilde{B}} y, v) \mid \forall y, v \in J_y \subseteq [0,1]\}$ is defined by:

$$^2R(x,u,y,v) = \{(x,u,y,v), \mu_R(x,u,y,v) \mid \forall (x,y,u,v) \in (x,J_x) \times (y,J_y)\} \quad (6)$$

where $x \in X$, $u \in J_x \subseteq [0, 1]$ and $y \in Y$, $v \in J_y \subseteq [0, 1]$.

Definition 5: A Type-1 Mamdani implication relation's membership $\mu_R(x, y)$ for the rule: *If x is A, then y is B*, is given by:

$$\mu_R(x,y) = \underset{\substack{x \in X \\ y \in Y}}{Min}(\mu_A(x), \mu_B(y)) \quad (7)$$

Definition 6: A Type-2 Mamdani implication relation's membership $\mu_R(x, u, y, v)$ for the rule: *If x is* $\widetilde{A}$*, then y is* $\widetilde{B}$, is given by:

$$\mu_R(x,u,y,v) = \underset{\substack{x \in X, u \in J_x \\ y \in Y, v \in J_y}}{Min} [\mu_{\widetilde{A}}(x,u), \mu_{\widetilde{B}}(y,v)] \quad (8)$$

where $J_x \in [0, 1]$ and $J_y \in [0, 1]$.

III. GENERALIZED MODUS PONENS

Given a fuzzy production rule: *If x is A, Then y is B* and a fuzzy fact *x is A'* where $A' \approx A$, we infer *y is B'* ($B' \approx B$). This is known as Generalized (Fuzzy) Modus Ponens [21-23]. We now present the Generalize Modus Ponens for T1 and T2 fuzzy logic.

A. Type-1 Modus Ponens

In T1 fuzzy logic, the membership distribution for $\mu_{B'}(y)$ is evaluated as follows:

$$\mu_{B'}(y) = \underset{x}{Max}[Min(\mu_{A'}(x), \mu_R(x,y))] \quad (9)$$

In discrete form, equation (9) can be written as:

$$B' = A' \circ ({}^1R) \quad (10)$$

where B' and A' are $(1 \times n)$ vectors, whose i^th^ components denote $\mu_{B'}(y)$ at at $x = x_i$ and $\mu_{A'}(x)$ at $x = x_i$ respectively for ordered $x = x_1, x_2, \ldots, x_n$ and $y = y_1, y_2, \ldots, y_n$. ${}^1R_i(x, y)$ relational matrix whose $(i, j)^{th}$ component denotes $\mu_R(x, y)$ at $x = x_i$ and $y = y_i$.

B. Type-2 Modus Ponens

Suppose, we are given $\mu_R(x, u, y, v)$ and $\mu(x, u')$, $u' = \mu_{\widetilde{A}'}(x)$, $\widetilde{A}' = \widetilde{A}$, say. Then, by Fuzzy Modus Ponens, we can infer *y is* $\widetilde{B}'$ with both secondary and primary membership distribution of the consequent clause.

B.1 *Secondary Membership Evaluation*

The secondary membership is given by:

$$\mu(y,v) = Max[\underset{\substack{x \in X \\ u, u' \in J_x}}{Min} (\mu(x,u'), \mu_R(x,u,y,v))] \quad (11)$$

In discrete form, equation (11) can be written as:

$$Q' = P' \circ ({}^2R) \quad (12)$$

where Q' and P' denote consequent and antecedent secondary membership vector and ${}^2R$ is a $(m \times q)$ relational matrix constructed from $\mu_R(x, u, y, v)$. The elements in P' denote the ordered memberships of (x, u') at

$$(x_1,u_1^1{}'),(x_1,u_2^1{}'),\ldots,(x_1,u_p^1{}'),$$
$$(x_2,u_1^2{}'),(x_2,u_2^2{}'),\ldots,(x_2,u_p^2{}'),\ldots,$$
$$(x_n,u_1^n{}'),(x_n,u_2^n{}'),\ldots,(x_n,u_p^n{}')$$

where $u_j^i \in J_x$ for $i \in [1, n]$ and $j \in [1, p]$ and $x \in \{x_1, \ldots, x_n\}$. The elements in Q' denote the ordered memberships of (y, v') at

$$(y_1,v_1^1{}'),(y_1,v_2^1{}'),\ldots,(y_1,v_p^1{}'),$$
$$(y_2,v_1^2{}'),(y_2,v_2^2{}'),\ldots,(y_2,v_p^2{}'),\ldots,$$
$$(y_n,v_1^n{}'),(y_n,v_2^n{}'),\ldots,(y_n,v_p^n{}')$$

where $v_j^i \in J_y$ for $i \in [1, n]$ and $j \in [1, p]$ and $y \in \{y_1, \ldots, y_n\}$. We can write ${}^2R = P'^T \circ Q'$ with $P' = P$ and $Q' = Q$, where P and Q denote vectors corresponding to secondary membership distribution of *x is A* and *y is B* respectively and T over a vector denotes its transposition.

B.2 Primary Membership Evaluation

To determine the primary membership distribution of the consequent clause *y is B'* , we first consider all the given *m* embedded fuzzy sets in the domain (DOM) of footprint of uncertainty (FOU) [6] of the antecedent fuzzy set $\widetilde{A}'$, where

$$FOU(\widetilde{A}') = \bigcup_{x \in X} J_x \quad \text{and} \quad DOM[FOU(\widetilde{A}')] = \bigcup_{i=1}^{m} \widetilde{A}_i,$$

where $\widetilde{A}_i$ denotes the i^th^ embedded antecedent fuzzy set. Suppose, we are given $R_i(x, y)$ that denotes the i^th^ relational matrix for mapping from x is $\widetilde{A}_i$ to y is $\widetilde{B}_i$, where $\widetilde{A}_i$ and $\widetilde{B}_i$ are i[th] embedded fuzzy sets in the universes of X and Y respectively. For computing the primary distribution of y is $B_i^{/}$ we use equation (9),

$$\mu_{\widetilde{B}_i'}(y) = \underset{x}{Max}[Min(\mu_{A_i'}(x), \mu_{R_i}(x,y))] \tag{13}$$

In discrete form, equation (13) can be written as:

$$B_i' = A_i' \circ ({}^1R_i) \tag{14}$$

where ${}^1R_i(x, y)$ is the relational matrix corresponding to the relation $\mu_{Ri}(x, y)$. Note that for p antecedent fuzzy sets $\widetilde{A}_1$, $\widetilde{A}_2$, …, $\widetilde{A}p$, we now have p consequent fuzzy sets $\widetilde{B}_1$, $\widetilde{B}_2$, …, $\widetilde{B}p$. The domain (DOM) of footprint of uncertainty (FOU) of the consequent fuzzy set $\widetilde{B}'$ is $DOM[FOU(\widetilde{B}')] = \bigcup_{i=1}^{p} \widetilde{B}_i'$. In a more expanded form, $DOM[FOU(\widetilde{B}')]$ looks like:

$$\begin{aligned}
&\{\mu_{\widetilde{B}_1'}(y_1) \mid y_1, \mu_{\widetilde{B}_2'}(y_1) \mid y_1 \ldots, \mu_{\widetilde{B}_p'}(y_1) \mid y_1,\\
&\mu_{\widetilde{B}_1'}(y_2) \mid y_2, \mu_{\widetilde{B}_2'}(y_2) \mid y_2, \ldots, \mu_{\widetilde{B}_p'}(y_2) \mid y_2,\\
&\ldots, \mu_{\widetilde{B}_1'}(y_n) \mid y_n, \mu_{\widetilde{B}_2'}(y_n) \mid y_n, \ldots, \mu_{\widetilde{B}_p'}(y_n) \mid y_n\}\\
&= \{v_1^1{}' \mid y_1, v_2^1{}' \mid y_1 \ldots, v_p^1{}' \mid y_1,\\
&v_1^2{}' \mid y_2, v_2^2{}' \mid y_2, \ldots, v_p^2{}' \mid y_2, \ldots,\\
&v_1^n{}' \mid y_n, v_2^n{}' \mid y_n, \ldots, v_p^n{}' \mid y_n\}
\end{aligned} \tag{15}$$

B.3 Combining Primary and Secondary Memberships

It is to be noted that ${}^2R(x, u, y, v)$ is a relation over two sets $\{(x, u)\}$ and $\{(y, v)\}$ where the doublets (x, u) and (y, v) maintain a fixed order i.e.,

$$\begin{aligned}
&(x_1,u_1^1),(x_1,u_2^1),\ldots,(x_1,u_p^1),\\
&(x_2,u_1^2),(x_2,u_2^2),\ldots,(x_2,u_p^2),\ldots,\\
&(x_n,u_1^n),(x_n,u_2^n),\ldots,(x_n,u_p^n)
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&(y_1,v_1^1),(y_1,v_2^1),\ldots,(y_1,v_p^1),\\
&(y_2,v_1^2),(y_2,v_2^2),\ldots,(y_2,v_p^2),\ldots,\\
&(y_n,v_1^n),(y_n,v_2^n),\ldots,(y_n,v_p^n)
\end{aligned} \tag{17}$$

Thus, the relational matrix ${}^2R(x, u, y, v)$ maps $\{(x, u)\}$ to $\{(y, v)\}$. Consequently, the secondary membership distribution μ(y, v), at $y = y_1, y_2, \ldots, y_n$ and $v = v_j^i$, $i \in [1, n]$ and $j \in [1, p]$ maintains the same order of (y, v) as introduced in (17).

If $\bigcup_{i=1}^{p} \widetilde{B}_i$ is expressed as in (15), then we obtain the same ordering of (y, v) in both (15) and (17). So, for each position-wise element of (y, v) in (17), we have a corresponding position-wise element in $\mu(y, v)$, representing the secondary distribution of the former element. Thus, we can write:

$$\begin{aligned}
\mu(y,v') = \{&\mu(y_1,v_1^1{}') \mid (y_1,v_1^1{}'), \mu(y_1,v_2^1{}') \mid (y_1,v_2^1{}') \ldots \mu(y_1,v_p^1{}') \mid (y_1,v_p^1{}'),\\
&\mu(y_2,v_1^2{}') \mid (y_2,v_1^2{}'), \mu(y_2,v_2^2{}') \mid (y_2,v_2^1{}'), \ldots \mu(y_2,v_p^2{}') \mid (y_2,v_p^2{}'), \ldots,\\
&\mu(y_n,v_1^n{}') \mid (y_n,v_1^n{}'), \mu(y_n,v_2^n{}') \mid (y_n,v_2^n{}') \ldots \mu(y_n,v_p^n{}') \mid (y_n,v_p^n{}')\}
\end{aligned} \tag{18}$$

IV. FUZZY ABDUCTIVE REASONING

In abductive reasoning, we are given a rule: *If x is A, then y is B* and a fuzzy fact *y is* $B^{/}$ *(*$B^{/} \approx B$*)*. From the rule and the fact, we infer *x is* $A^{/}$ *(*$A^{/} \approx A$*)*.

A. Type-1 Fuzzy Abduction

In T1 fuzzy abduction, we are given the membership distribution of y is $B^{/}$ i.e. $\mu_{B'}(y)$ and the relational matrix *R(x, y)* for the implication function: *If x is A, then y is B*. We can derive $\mu_{A'}(x)$ by:

$$\mu_{A'}(x) = \underset{y \in Y}{Max}[Min(\mu_{B'}(y), R(x,y))] \tag{19}$$

which in discrete form can be written as:

$$\begin{aligned}
A' &= B' \circ ({}^1R)^{-1}\\
&= B' \circ ({}^1R)^T
\end{aligned} \tag{20}$$

where $A^{/}$ and $B^{/}$ are $(1\times n)$ vectors, ${}^1R$ is a relational matrix, as introduced in section III and "∘" denotes the max-min operation. Here, for the sake of simplicity, we take $R^{-1} = R^T$ [19] for a fuzzy relational matrix R.

B. Type-2 Fuzzy Abduction

In T-2 fuzzy abduction, we need to evaluate both the primary and the secondary memberships of the antecedent clause *x is* $A^{/}$. This is a three-step procedure as outlined in the algorithm below using our existing notations.

Algorithm: Fuzzy Type-2 Abduction

Input: $Q^{/}$, ${}^2R$, $B_i^{/}$, ${}^1R_i$ $\forall i$

Output: $P^{/}$, $A_i^{/}$ $\forall i$

Begin

1. Secondary Membership evaluation

Compute secondary membership distribution vector $P^{/}$ of the antecedent clause by:

$$P' = Q' \circ ({}^2R)^{-1} = Q' \circ ({}^2R)^T$$ (follows from equation (12))

2. Primary Membership evaluation

Compute primary membership distribution vectors A_i' $\forall i$ by:

$$A_i' = B_i' \circ ({}^1R_i)^{-1} = B_i' \circ ({}^1R_i)^T$$ (follows from equation (14))

Construct a vector $\tilde{A}'$ by assembling elements of $\tilde{A}_1'$, $\tilde{A}_2'$, …, $\tilde{A}_p'$ in the following specific order:

$$\tilde{A}' = [\mu_{\tilde{A}_1'}(x_1) \mid x_1, \mu_{\tilde{A}_2'}(x_1) \mid x_1 \ldots, \mu_{\tilde{A}_p'}(x_1) \mid x_1,$$
$$\mu_{\tilde{A}_1'}(x_2) \mid x_2, \mu_{\tilde{A}_2'}(x_2) \mid x_2, \ldots, \mu_{\tilde{A}_p'}(x_2) \mid x_2, \ldots,$$
$$\mu_{\tilde{A}_1'}(x_n) \mid x_n, \mu_{\tilde{A}_2'}(x_n) \mid x_n \ldots, \mu_{\tilde{A}_p'}(x_n) \mid x_n]$$
$$= \{u_1' \mid x_1, \ldots, u_p' \mid x_1, \ldots, u_1' \mid x_n, \ldots, u_p' \mid x_n\} \quad (21)$$

3. Combination of primary and secondary distributions

Since vector P' includes the secondary membership in a specific order like the order of $(u^{i'}_j, x_i)$ in $\tilde{A}'$, we construct $\mu(x, u')$ by picking up position-wise elements from $\tilde{A}'$ and P' respectively, where

$$\mu(x,u') = \{\mu(x_1,u_1^1{}') \mid (x_1,u_1^1{}'), \mu(x_1,u_2^1{}') \mid (x_1,u_2^1{}'),$$
$$\ldots, \mu(x_1,u_p^1{}') \mid (x_1,u_p^1{}'),$$
$$\mu(x_2,u_1^2{}') \mid (x_2,u_1^2{}'), \mu(x_2,u_2^2{}') \mid (x_2,u_2^2{}'),$$
$$\ldots, \mu(x_2,u_p^2{}') \mid (x_2,u_p^2{}'), \ldots,$$
$$\mu(x_n,u_1^n{}') \mid (x_n,u_1^n{}'), \mu(x_n,u_2^n{}') \mid (x_n,u_2^n{}')$$
$$\ldots, \mu(x_n,u_p^n{}') \mid (x_n,u_p^n{}')\}$$

End.

V. EXAMPLES

In this section, we demonstrate computation of T2 and T1 relational matrices, T2 fuzzy modus ponens and T2 fuzzy abduction. Example 1 deals with construction of relational matrices ${}^2R$ and ${}^1R$'s. Examples 2.1 and 2.2 respectively illustrate T2 modus ponens and T2 abduction. In all these examples, we consider the following two membership distributions: *μ(x, u)* and *μ(y, v)* for the rule: *If x is A, then y is B.* Given,

$\mu(x, u) = \{0.1|(x_1, 0.2), 0.3|(x_1, 0.4), 0.5|(x_1, 0.5),$
$0.2|(x_2, 0.1), 0.4|(x_2, 0.3), 0.8|(x_2, 0.6),$
$0.3|(x_3, 0.1), 0.5|(x_3, 0.4), 0.8|(x_3, 0.8)\}$ (22)

$\mu(y, v) = \{0.2|(y_1, 0.3), 0.4|(y_1, 0.4), 0.5|(y_1, 0.6),$
$0.4|(y_2, 0.1), 0.5|(y_2, 0.3), 0.7|(y_2, 0.8),$
$0.2|(y_3, 0.2), 0.7|(y_3, 0.4), 0.9|(y_3, 0.7)\}$ (23)

Example 1: *Relational Matrix Computation*

From equations (22) and (23), we obtain the secondary membership vectors **P** and **Q** as indicated.

	$(x_1, 0.2)$	$(x_1, 0.4)$	$(x_1, 0.5)$	$(x_2, 0.1)$	$(x_2, 0.3)$	$(x_2, 0.6)$	$(x_3, 0.1)$	$(x_3, 0.4)$	$(x_3, 0.8)$
P =	[0.1	0.3	0.5	0.2	0.4	0.8	0.3	0.5	0.8]

	$(y_1, 0.3)$	$(y_1, 0.4)$	$(y_1, 0.6)$	$(y_2, 0.1)$	$(y_2, 0.3)$	$(y_2, 0.8)$	$(y_3, 0.2)$	$(y_3, 0.4)$	$(y_3, 0.7)$
Q =	[0.2	0.4	0.5	0.4	0.5	0.7	0.2	0.7	0.9]

From the same equations (22) and (23), we obtain the embedded fuzzy sets $\tilde{A}_i$ and $\tilde{B}_i$ for i=1, 2 and 3:

$$\tilde{A}_1 = \{0.2 \mid x_1, 0.1 \mid x_2, 0.1 \mid x_3\}$$
$$\tilde{A}_2 = \{0.4 \mid x_1, 0.3 \mid x_2, 0.4 \mid x_3\}$$
$$\tilde{A}_3 = \{0.5 \mid x_1, 0.6 \mid x_2, 0.8 \mid x_3\} \quad (24)$$
$$\tilde{B}_1 = \{0.3 \mid y_1, 0.1 \mid y_2, 0.2 \mid y_3\}$$
$$\tilde{B}_2 = \{0.4 \mid y_1, 0.3 \mid y_2, 0.4 \mid y_3\}$$
$$\tilde{B}_3 = \{0.6 \mid y_1, 0.8 \mid y_2, 0.7 \mid y_3\}$$

The corresponding vectors of the above fuzzy sets are given below:

	x_1	x_2	x_3
A₁ =	[0.2	0.1	0.1]

	x_1	x_2	x_3
A₂ =	[0.4	0.3	0.4]

	x_1	x_2	x_3
A₃ =	[0.5	0.6	0.8]

(25)

	y_1	y_2	y_3
B₁ =	[0.3	0.1	0.2]

	y_1	y_2	y_3
B₂ =	[0.4	0.3	0.4]

	y_1	y_2	y_3
B₃ =	[0.6	0.8	0.7]

We now obtain ${}^2R = P^T \circ Q$ which is given by:

Table 1. T2 Relational Matrix

(x, u) \ (y, v)	$(y_1, .3)$	$(y_1, .4)$	$(y_1, .6)$	$(y_2, .1)$	$(y_2, .3)$	$(y_2, .8)$	$(y_3, .2)$	$(y_3, .4)$	$(y_3, .7)$
$(x_1, 0.2)$	.1	.1	.1	.1	.1	.1	.1	.1	.1
$(x_1, 0.4)$	.2	.3	.3	.3	.3	.3	.2	.3	.3
$(x_1, 0.5)$	.2	.4	.5	.4	.5	.5	.2	.5	.5
$(x_2, 0.1)$	.2	.2	.2	.2	.2	.2	.2	.2	.2
$(x_2, 0.3)$	.2	.4	.4	.4	.4	.4	.2	.4	.4
$(x_2, 0.6)$	.2	.4	.5	.4	.5	.7	.2	.7	.8
$(x_3, 0.1)$	.2	.3	.3	.3	.3	.3	.2	.3	.3
$(x_3, 0.4)$	.2	.4	.5	.4	.5	.5	.2	.5	.5
$(x_3, 0.8)$	.2	.4	.5	.4	.5	.7	.2	.7	.8

For the present example, we have three primary relational matrices. The three matrices are computed below:

$${}^1R_1 = A_1^T \circ B_1 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{ccc} y_1 & y_2 & y_3 \\ \hline 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{array}$$

$$
{}^1R_2 = A_2^T \circ B_2 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{ccc} y_1 & y_2 & y_3 \\ \left(\begin{array}{ccc} 0.4 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.4 \end{array}\right) \end{array}
$$

$$
{}^1R_3 = A_3^T \circ B_3 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{ccc} y_1 & y_2 & y_3 \\ \left(\begin{array}{ccc} 0.5 & 0.5 & 0.5 \\ 0.6 & 0.6 & 0.6 \\ 0.6 & 0.8 & 0.7 \end{array}\right) \end{array}
$$

Example 2: *T2 Fuzzy Modus Ponens*
Given $P^{/}= P$ and ${}^2R$, we first evaluate $Q^{/}$. Next, we evaluate $B^{/}$ from known $A_i^{/}$ and ${}^1R_i$ for i = 1, 2 and 3. We then combine $Q^{/}$ with $B^{/}$ to obtain $\mu(y, v)$.

$$
\begin{aligned}
\mathbf{Q}^{/} &= \mathbf{P}^{/} \circ {}^2R \\
&= [0.2 \quad 0.4 \quad 0.5 \quad 0.4 \quad 0.5 \quad 0.7 \quad 0.2 \quad 0.8 \quad 0.8]
\end{aligned} \qquad (26)
$$

$$
\mathbf{B_1}^{/} = \mathbf{A_1}^{/} \circ {}^1R_1 = \begin{array}{ccc} y_1 & y_2 & y_3 \\ [0.2 & 0.1 & 0.2] \end{array}
$$

$$
\mathbf{B_2}^{/} = \mathbf{A_2}^{/} \circ {}^1R_2 = \begin{array}{ccc} y_1 & y_2 & y_3 \\ [0.4 & 0.3 & 0.4] \end{array}
$$

$$
\mathbf{B_3}^{/} = \mathbf{A_3}^{/} \circ {}^1R_3 = \begin{array}{ccc} y_1 & y_2 & y_3 \\ [0.6 & 0.8 & 0.7] \end{array}
$$

We now combine $\mathbf{B}^{/} = \mathbf{B_1}^{/} \cup \mathbf{B_2}^{/} \cup \mathbf{B_3}^{/}$

$$
= \begin{array}{ccccccccc} y_1 & y_1 & y_1 & y_2 & y_2 & y_2 & y_3 & y_3 & y_3 \\ [0.2 & 0.4 & 0.6 & 0.1 & 0.3 & 0.8 & 0.2 & 0.4 & 0.7] \end{array} \qquad (27)
$$

Combining (26) and (27), we obtain

$$
\begin{aligned}
\mu(y, v^{/}) = \{&0.2|\,(y_1, 0.2),\ 0.4|\,(y_1, 0.4),\ 0.5|\,(y_1, 0.6), \\
&0.4|\,(y_2, 0.1),\ 0.5|\,(y_2, 0.3),\ 0.7|\,(y_2, 0.8), \\
&0.2|\,(y_3, 0.2),\ 0.8|\,(y_3, 0.4),\ 0.8|\,(y_3, 0.7)\}
\end{aligned} \qquad (28)
$$

Example 3: *T2 Fuzzy Abduction*
Given $\mathbf{Q}^{/}=\mathbf{Q}$ (say) and ${}^2R$, we first obtain $\mathbf{P}^{/}$. We then compute $\mathbf{A_i}^{/}$ from $\mathbf{B_i}^{/}$ and ${}^1R_i$ for i=1 to 3. We next obtain $\mathbf{A}^{/}$ from the union of $\mathbf{A_i}^{/}$s. Lastly, we combine $\mathbf{A}^{/}$ with $\mathbf{P}^{/}$ to obtain $\mu(x, u^{/})$. Detailed computations are shown below:

$$
\begin{aligned}
\mathbf{P}^{/} &= \mathbf{Q}^{/} \circ {}^2R^T \\
&= [0.1 \quad 0.3 \quad 0.5 \quad 0.2 \quad 0.4 \quad 0.8 \quad 0.3 \quad 0.5 \quad 0.8]
\end{aligned} \qquad (29)
$$

The primary membership distributions of the antecedent clause can be obtained as

$$
\begin{aligned}
\mathbf{A_1}^{/} &= \mathbf{B_1}^{/} \circ ({}^1R_1)^T = \begin{array}{ccc} x_1 & x_2 & x_3 \\ [0.2 & 0.1 & 0.1] \end{array} \\
\mathbf{A_2}^{/} &= \mathbf{B_2}^{/} \circ ({}^1R_2)^T = \begin{array}{ccc} x_1 & x_2 & x_3 \\ [0.4 & 0.3 & 0.4] \end{array} \\
\mathbf{A_3}^{/} &= \mathbf{B_3}^{/} \circ ({}^1R_3)^T = \begin{array}{ccc} x_1 & x_2 & x_3 \\ [0.5 & 0.6 & 0.8] \end{array}
\end{aligned} \qquad (30)
$$

We now compute $\mathbf{A}^{/} = \mathbf{A_1}^{/} \cup \mathbf{A_2}^{/} \cup \mathbf{A_3}^{/}$

$$
= \begin{array}{ccccccccc} x_1 & x_1 & x_1 & x_2 & x_2 & x_2 & x_3 & x_3 & x_3 \\ [0.2 & 0.4 & 0.5 & 0.1 & 0.3 & 0.6 & 0.1 & 0.4 & 0.8] \end{array} \qquad (31)
$$

Combining (29) and (31), we can write the secondary membership values of the antecedent clause $\mu(x, u^{/})$:

$$
\begin{aligned}
\mu(x, u^{/}) = \{&0.1|\,(x_1, 0.2),\ 0.3|\,(x_1, 0.4),\ 0.5|\,(x_1, 0.5), \\
&0.2|\,(x_2, 0.1),\ 0.4|\,(x_2, 0.3),\ 0.8|\,(x_2, 0.6), \\
&0.3|\,(x_3, 0.1),\ 0.5|\,(x_3, 0.4),\ 0.8|\,(x_3, 0.8)\}.
\end{aligned}
$$

VI. PERFORMANCE EVALUATION

For performance evaluation, we start with a given $P^{/}$, and evaluate $Q^{/}$ by fuzzy modus ponens and then evaluate $P^{//}$ from $Q^{/}$ by fuzzy abduction. We also evaluate $B^{/}$ from a given $A^{/}$ (corresponding to type-2 fuzzy set $\widetilde{A}'$) by fuzzy modus ponens and further evaluate $A^{//}$ from $B^{/}$ by fuzzy abduction. We now design a performance metric E, given by:

$$
E = \|P' - P''\| + \|A' - A''\| \qquad (32)
$$

where ||.|| denotes the Euclidean norm.

It may be verified from examples 1, 2.1 and 2.2 that the error norm E (computed from equation (32)) is found to be zero. That is, when $A^{/}=A$ and $P^{/}=P$ for the rule: *If x is A, then y is B*, we obtain $Q^{/}=Q$ and $B^{/}=B$ by fuzzy modus ponens. Further, when we evaluate $P^{//}$ and $A^{//}$ from $Q^{/}$ and $B^{/}$ respectively, we obtain $P^{//} = P^{/} = P$ and $A^{//} = A^{/} = A$. This signifies the basis of the error function E.

VII. CONCLUSIONS AND FUTURE SCOPE

The paper addresses an important problem of abductive reasoning using type-2 fuzzy sets. A novel algorithm is proposed as a solution. Both primary and secondary membership functions for the antecedent clause are computed, when the same for the consequent clause are known. The results of computation support the local basis of abductive reasoning. A performance metric is designed and the performance of the T2 fuzzy abduction algorithm with respect to the proposed metric is analyzed.

Future work includes further performance improvement of the discussed algorithm with a better alternative for the inverse computation [18-20]. As a part of future work, we would also focus on the extension of the proposed T2 fuzzy abduction to i) single rule with multiple clauses and ii) multiple rules with single/multiple clauses.

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