Extending the Contraposition Property of Propositional Logic for Fuzzy Abduction

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***Abstract*—Abduction deals with assumption-based reasoning to explain an observation. In the context of fuzzy reasoning, abduction attempts to determine the membership function of the fuzzy propositions present in the antecedent of a rule when the membership functions for the propositions in the consequent of the rule are given. Currently available models of fuzzy abduction are capable of inferring the membership function of the antecedent clause accurately when the antecedent includes single fuzzy proposition. However, when the antecedent clause of a rule contains multiple fuzzy propositions, these models fail to determine the independent membership function of the individual propositions present in the antecedent. This paper presents a new formulation to handle the above problem by fuzzy extension of the well-known contraposition property of propositional logic. Several interesting properties due to the fuzzy extension of the classical contraposition have been derived. An algorithm for automated abduction using the extended contraposition property has been developed to demonstrate the principle of abduction with rules containing one or more fuzzy propositions in the antecedent/consequent. The time complexity of the proposed fuzzy abduction for a sequence of n-chained rules, where each rule has m fuzzy propositions, is $O(mn)$, considering a uniform cost for composition operation and t-norm computation of the antecedent.**

***Index Terms*—Contraposition property, fuzzy abduction, fuzzy logic network (FLN), implication relations, propositional logic.**

I. Introduction

ABDUCTION refers to an assumption-based reasoning strategy to explain an observation [3]. The principle of abduction aims to determine a subset from a given set of assumed premises, which, in conjunction with a given knowledge base, is capable of explaining the observation. For example, in a diagnostic problem, we need to identify a defective component from a list of possible defects $d_1, d_2, \ldots, d_n$ to support a set of observations about malfunctioning of the system using a given set of diagnostic rules.

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Several attempts have been made to handle the abduction problem [4], [6]–[12], [14]–[17], [21]–[23], [28], [29], [31], [32], [35], [37], [39]–[42]. Most of the techniques on abduction can be broadly classified into two major heads: causal [19] and implicational [1] reasoning. Although treated similarly in our regular use, causality and implication are different in the context of reasoning, as the former requires a temporal ordering between the antecedent and the consequent, while the latter does not require such ordering [45]. In a simplistic causal reasoning [46] problem, P is a cause of Q, Q is observed, and abduction refers to inferring P. Researchers employed probabilistic frameworks to handle the causal abduction. One of the well-known works in this regard is the belief revision paradigm in a Bayesian network. Pearl's belief revision model [47] in a causal framework is possibly the first successful work in this regard. Several extensions to Pearl's model have been undertaken, each with a new flavor [7], [8], [10].

Similarly, in implicational reasoning, abduction refers to extraction of P from the rule: if P then Q, and the observation Q. As the logic of propositions/predicates is incompetent to handle abduction [48], the early works on implicational abduction started with nonstandard logic. These logics fundamentally differ from their classical first-order (propositional/predicate logic) [49] counterpart by their inherent capability of nonmonotonic [50], [51] reasoning. It may be added here that an inferred statement in the context of nonmonotonic reasoning may sometimes contradict its predecessors/premises. Unless such contradiction is detected, we can infer P from the fact "Q" and the given rule "if P then Q." In case a contradiction is found between the inferred P and the given/inferred set of statements, P is withdrawn. For example, given a default rule "if x is a bird, then x can fly" and an observation "x flies" makes someone guess that x is a bird. This assumption-based default abduction [30] is valid until it is discovered (by other defaults or through observations) that x is a bat (and not a bird). Several nonmonotonic and default logic models [29], [52], [53], [61] have been employed to handle abduction. Modal logic is also considered in the literature of abduction [21].

The logic of fuzzy sets [44] offers interesting solutions to the principles of abduction [2], [38]. In general, fuzzy abduction attempts to infer the membership function (MF) [62] of the antecedent, when the MFs of the consequent and the implication relation between the antecedent and the consequent of a rule are known. One primitive approach to construct a solution to this problem is to design relational equations involving known MF of the consequent and unknown antecedent and then to provide a relational algebraic approach to solve the problem. The abduction problem thus is translated into fuzzy relational equations. The early works on fuzzy abduction employed transpose the implicational relation to derive fuzzy premises of a given rule [18], [19]. The transpose-based solution is advantageous for its simplicity in approach and insignificant computational complexity. However, the quality of transpose-based solution is not always acceptable [36]. Recently, researchers, instead of directly solving the fuzzy relational equation [24]–[27], employ a heuristic algorithm to compute max-min compositional inverse fuzzy relations [36] and offer interesting solutions to fuzzy abduction using the inverse relation. More recently, an alternative approach to computing the optimal inverse (fuzzy) implicational relation is proposed in [11] in connection with the abduction problem. This approach [11] seems to have less complexity than the approach in [36] and thus can effectively be used for abduction.

Here, we propose an alternative approach to handle the fuzzy abduction problem using an extension of the well-known contraposition property of propositional logic. For the sake of convenience, it may be added here that transforming a given rule "*if p then q*" to its equivalent form "*if* $\neg q$ *then* $\neg p$" is the basis of propositional contraposition. When the antecedent is a conjunction of atomic propositions $p_1, p_2, \ldots, p_n$ and the consequent is a disjunction of atomic propositions $q_1, q_2, \ldots, q_m$, we transform a given primal rule to its equivalent dual form with modified antecedent as the conjunction of $\neg q_j$'s and modified consequent as the disjunction of $\neg p_i$'s [64]. This transformation has been referred to as the *extended contraposition property*. The transformed rule asserts falsehood of at least one p_i, given all the q_j's are false. It may be noted that the transformed rule is an explicit representation of the propositional backward chaining principle [55] applied on the original rule.

In this paper, we study the scope of fuzzy abduction by extending the contraposition property further in the context of fuzzy logic. We have shown that the extended contraposition property holds good in fuzzy logic, if the implication used in the fuzzy production rules is realized by Diens–Rescher- or Lukasiewicz-type implication functions. To illustrate the fuzzy extension of the contraposition property, consider x and y as two linguistic variables with universes U and V, respectively. Let A and B be fuzzy subsets of U and V, respectively; in other words, A and B are linguistic values of x and y, respectively. The contraposition property in the context of fuzzy logic transforms a primal rule "if x is A, then y is B" into its equivalent dual form "if y is $\overline{B}$, then x is $\overline{A}$," where the bar above a fuzzy set denotes its complement. Therefore, if we know the MF of y is B' ($B' \approx B$), we can derive the MF of x is A' ($A' \approx A$) using the dual rule.

It may be mentioned here that *generalized modus tollens* (GMT)-based fuzzy abduction with a primal rule "if x_1 is A_1 and x_2 is A_2, then y_1 is B_1 or y_2 is B_2," where the symbols have standard meaning, returns a single relation involving t-norm of the MFs of the premises x_i is A'_i for all i together, when the MFs of y_j is B'_j for all j are given. Therefore, computation of the individual MF of x_i is A'_i for all i separately becomes an indeterminate task in GMT-based fuzzy abduction. The fuzzy extension of the propositional contraposition property overcomes this limitation, as the conjunction of the premises, x_i is A_i, of the primal rule now appears as disjunction of their complements in the consequent part of the dual rule. Therefore, given the MF of y_j is $\overline{B'_j}$ for all j, we can evaluate the MF for x_i is $\overline{A'_i}$ for all i by a generalized (fuzzy) modus ponens. Such abductive interpretation has applications in many engineering problems, such as diagnosis and historical time-series prediction [33], [34].

A question naturally arises: How good is contraposition-based fuzzy abduction. To study the qualitative performance of the proposed fuzzy abduction, we consider retrieval of MF as an important measure. Retrieval here means that the computed MF x_i is A'_i will be equal to the given MF x_i is A_i of the antecedent for all i, when the observed MF y_j is B'_j and the given MF y_j is B_j of the consequent are equal for all j. Retrieval, however, requires satisfaction of specific conditions involving the MFs of x_i is A_i, y_j is B_j, and y_j is B'_j. The conditions for abductive retrieval for Diens–Rescher- and Lukasiewicz-based implication have been derived here.

Reasoning through chained sequence of rules is an important issue in fuzzy logic [43]. This has been undertaken in this paper first by representing the chained rules as a fuzzy inferential network and then by transforming the rules in the network from their primal to dual form using extended contraposition property. Given the observed MF of the terminal [54] fuzzy propositions in the primal network, the principle of abduction can be applied in a layerwise fashion in the dual network, starting computation in a layer with known MFs and continuing computation in a layerwise manner until the MFs of the desired propositions are obtained. An algorithm for automated abduction in a fuzzy logic network (FLN) is presented following the principle outlined above. Reasoning efficiency [54] of the proposed algorithm depends greatly on the number of chained rules and the number of antecedent clauses in the rules. For an n-chain ruled system with m fuzzy propositions per rule, the time complexity of the algorithm is $O(mn)$.

The rest of this paper is organized into five sections. In Section II, we extend the contraposition property of propositional logic and construct a formulation of fuzzy abduction using the extension. In Section III, we derive the conditions for abductive retrieval for the Lukasiwicz- and Diens–Rescher-type implication functions. Here, we also derive the condition for abductive retrieval when the formulation is undertaken as GMT. Section IV provides an algorithm for automated abductive reasoning with multiple chained rules using the dual FLN. Conclusions are listed in Section V.

II. Contraposition-Based Fuzzy Abduction

In this section, we study two important fuzzy extensions of propositional properties: 1) decomposition and 2) contraposition.

A. Fuzzy Decomposition

Let p, q, r, and s be four atomic (binary valued) propositions. In propositional logic, the rule "$p \wedge q \to r \vee s$" can equivalently be expressed as two rules "$p \wedge q \to r; p \wedge q \to s$," where $\wedge$, $\vee$, and $\to$ denote AND, OR, and implication operators, respectively. We would now like to examine whether the above property holds in the logic of fuzzy sets.

Let X_i and Y_j be universes of fuzzy linguistic variables x_i and y_j, respectively, for $i = 1$ to n and $j = 1$ to m. Let A_i and

B_j be fuzzy sets in X_i and Y_j, respectively. Consider a fuzzy (primal) rule:

Rule 1: If x_1 is A_1 and x_2 is A_2 and ... and x_n is A_n then y_1 is B_1 or y_2 is B_2 or ... or y_m is B_m.

We would now present one interesting decomposition property of the rule, by which the rule would be represented as m subrules connected by OR operators, as indicated below.

Rule 2: (If x_1 is A_1 and x_2 is A_2 and ... and x_n is A_n, then y_1 is B_1) or

(If x_1 is A_1 and x_2 is A_2 and ... and x_n is A_n, then y_2 is B_2) or

... ...

or (If x_1 is A_1 and x_2 is A_2 and ... and x_n is A_n, then y_m is B_m).

Equivalence of any two fuzzy rules can be established by testing equality of their fuzzy implication relations [27]. We would now examine the equivalence of Rules 1 and 2 using Diens–Rescher and Lukasiewicz implication functions. Let t and s be scalar t- and s-norm operators [18]. The t-norm (here, min) and s-norm (here, max) between two membership values $\mu_{A_i}(x_i) = a_i$ and $\mu_{B_j}(y_j) = b_j$ are given by

$$\mu_{A_i}(x_i) t\, \mu_{B_j}(y_j) = \{a_i \wedge b_j\}$$
$$\mu_{A_i}(x_i) s\, \mu_{B_j}(y_j) = \{a_i \vee b_j\}.$$

While computing the t- or s-norms of two MFs, a specific order of the indices i and j is to be considered. Here, for each integer i, we consider all j in ascending order and then consider the next integer i in ascending order until $i = n$ and $j = m$. The cumulative t- norm of three or more MFs $\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_n}$, where μ_{A_i} is defined on X_i, yields an n-dimensional relation, say, R, given by $R = \mathop{T}\limits_{i=1}^{n}(A_i)$, where A_i is a vector of membership values representing a fuzzy set defined on X_i and an element of R is defined as

$$\begin{aligned} &R(x_1, x_2, \ldots, x_n | x_i \in X_i; i = 1, \ldots, n) \\ &\quad = (\mu_{A_1}(x_1) t\, \mu_{A_2}(x_2), \ldots, t\, \mu_{An}(x_n)) \\ &\quad = (\ldots((\mu_{A_1}(x_1) t\, \mu_{A_2}(x_2)) \ldots t\, \mu_{A_n}(x_n)) \ldots) \\ &\quad = \mathop{t}\limits_{i=1}^{n} \mu_{A_i}(x_i). \end{aligned} \quad (1)$$

Similarly, cumulative s-norm of $\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)$ is denoted $\mathop{S}\limits_{i=1}^{n} \mu_{A_i}(x_i)$. On the other hand, the cumulative s-norm among n MFs A_i, denoted by $\mathop{S}\limits_{i=1}^{n}(A_i)$, is an n-dimensional relation which can be evaluated by replacing t by s in (1).

We now define an S-norm operator between two relational matrices R_1 $(X_1, X_2, \ldots, X_n; Y_1)$ and R_2 $(X_1, X_2, \ldots, X_n; Y_2)$, which yields a new relational matrix R_{12} $(X_1, X_2, \ldots, X_n; Y_1, Y_2)$; the row indices of the resulting matrix is the joint occurrence of the valuation space of $x_1, x_2, \ldots, x_n$, where the column indices represent the joint occurrences of y_1, y_2. Let $R_{12}(x_1, x_2, \ldots, x_n; y_1, y_2), y_1, y_2 \in (Y_1 \times Y_2)$ be an element of R_{12}, which is obtained by applying an s-norm on R_1 $(x_1, x_2, \ldots, x_n;\, y_1)$ and R_2 $(x_1, x_2, \ldots, x_n;\, y_2)$, which are elements of R_1 and R_2, respectively. Thus

$$\begin{aligned} &R_{12}(x_1, x_2, \ldots, x_n; y_1, y_2) \text{ for } y_1, y_2 \in (Y_1 \times Y_2) \\ &= R_1(x_1, x_2, \ldots, x_n; y_1) s\, R_2(x_1, x_2, \ldots, x_n; y_2) \text{ for } y_1 \in Y_1 \end{aligned}$$

and $y_2 \in Y_2$; s is a scalar s-norm [18].

If the cardinalities of Y_1 and Y_2 are m_1 and m_2, respectively, then to define the entire relation R_{12}, for each possible $m_1 m_2$ pairs, the above s-norm operation is to be computed. For notational simplicity, we denote

$$\begin{aligned} &R_{12}(X_1, X_2, \ldots, X_n; Y_1, Y_2) = R_{12}, \text{ say} \\ &= \mathop{S}\limits_{\substack{y_1 \in Y_1 \\ y_2 \in Y_2}} (R_1(x_1, x_2, \ldots, x_n; y_1), (R_2(x_1, x_2, \ldots, x_n; y_2)). \end{aligned}$$

This notation will help us in proving some results later. We call S an S-norm over two relations, which is realized by applying the $\vee$ operator on the columns of R_1 and R_2 for y_1, $y_2 \in Y_1 \times Y_2$ in an ordered manner as defined above. Here, for a given y_1, the above S-norm operation is performed for all y_2 in a fixed order, considering a fixed instantiation of x_1, $x_2, \ldots, x_n$ in $X_1 \times X_2 \times \cdots \times X_n$ for both R_1 and R_2. Note that we can repeatedly apply the above S-norm to define R_{123} from R_{12} and R_3, and so on. We now provide an example to illustrate the computation of R_{12}.

Example 1: Given $X_1 = \{1, 2\}, X_2 = \{3, 4\}, Y_1 = \{5, 6\}$, and $Y_2 = \{7, 8\}$. Let $x_i \in X_i$ and $y_i \in Y_i$ for $i = 1$ to 2. Let $\mu_{A_1}(X_1) = \{0.1/1, 0.7/2\}, \mu_{A_2}(X_2) = \{0.3/3, 0.9/4\}, \mu_{B_1}(Y_1) = \{0.2/5, 0.8/6\}$, and $\mu_{B_2}(Y_2) = \{0.6/7, 0.9/8\}$.

We now construct relations R_1 and R_2 for the Rules 1 and 2, respectively, using the Lukasiewicz implication function, where Rule 1 is "*if* x_1 *is* A_1 *and* x_2 *is* A_2, *then* y_1 *is* B_1" and Rule 2 is "*if* x_1 *is* A_1 *and* x_2 *is* A_2, *then* y_2 *is* B_2.

$$R_1 = \begin{array}{c} x_1, x_2 \;\; y_1 \\ 1,3 \\ 1,4 \\ 2,3 \\ 2,4 \end{array} \begin{array}{c} \;\;5 \quad\;\; 6 \\ \begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 1.0 \\ 0.9 & 1.0 \\ 0.5 & 1.0 \end{bmatrix} \end{array}, \quad R_2 = \begin{array}{c} x_1, x_2 \;\; y_2 \\ 1,3 \\ 1,4 \\ 2,3 \\ 2,4 \end{array} \begin{array}{c} \;\;7 \quad\;\; 8 \\ \begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 1.0 \\ 1.0 & 1.0 \\ 0.9 & 1.0 \end{bmatrix} \end{array}.$$

Now, we construct $R_{12} = R_1\, S\, R_2$.

$$R_{12} = \begin{array}{c} x_1, x_2 \;\; y_1\, y_2 \\ 1,3 \\ 1,4 \\ 2,3 \\ 2,4 \end{array} \begin{array}{c} \quad 5,7 \qquad\quad 5,8 \qquad\quad 6,7 \qquad\quad 6,8 \\ \begin{bmatrix} (1.0s1.0) & (1.0s1.0) & (1.0s1.0) & (1.0s1.0) \\ (1.0s1.0) & (1.0s1.0) & (1.0s1.0) & (1.0s1.0) \\ (0.9s1.0) & (0.9s1.0) & (1.0s1.0) & (1.0s1.0) \\ (0.5s0.9) & (0.5s1.0) & (1.0s0.9) & (1.0s1.0) \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 0.9 & 1.0 & 1.0 & 1.0 \end{bmatrix}.$$

Note that R_{12}, in general, should be a 4-D relation of size $2 \times 2 \times 2 \times 2$. However, we represented it as a 2-D relation of size $4\,(=2 \times 2) \times 4\,(2 \times 2)$ for the sake of convenience in reasoning using max-min composition operation [19].

The S-norm of R_1 $(X_1, X_1, \ldots, ; Y_1)$ and R_2 $(X_1, X_1, \ldots, ; Y_2), \ldots, R_m$ $(X_1, X_1, \ldots, ; Y_m)$ is obtained in $(m - 1)$ steps with intermediate relation $R_{12}, R_{123}, \ldots, R_{123\ldots(m-1)}$, where

$$R_{12} = \underset{\substack{y_1 \in Y_1 \\ y_2 \in Y_2}}{S} \{(R_1(x_1, x_2, \ldots, x_n; y_1) \quad (R_2(x_1, x_2, \ldots, x_n; y_2)\}$$

$$R_{123} = \underset{\substack{y_1, y_2 \in Y_1 \times Y_2 \\ y_3 \in Y_3}}{S} \{R_{12}, R_3\}$$

$$R_{1234} = \underset{\substack{y_1 y_2 y_3 \in Y_1 \times Y_2 \times Y_3 \\ y_4 \in Y_4}}{S} \{R_{123}, R_4\}$$

and so on, and finally

$$R_{123\ldots m} = \underset{\substack{y_1 y_2 \ldots y_{(m-1)} \in Y_1 \times Y_2 \times \cdots \times Y_{m-1} \\ y_m \in Y_m}}{S} \{R_{123\ldots(m-1)}, R_m\}.$$

When $R_1, R_2, \ldots, R_m$ are matrix relations, we write $R_{123\ldots m} = R_1 S R_2 \ldots S R_m = \overset{m}{\underset{i=1}{S}} R_i$, say, where the computation of the successive S-norms is performed in order of their occurrence. Let $R\ (X_1, X_2, \ldots, X_n;\ Y_1, Y_2, \ldots, Y_m)$ and $R_i(X_1, X_2, \ldots, X_n; Y_i)$ be the implication relations for Rule 1 and subrule *i* under Rule 2, respectively. Theorem 1 indicates that $R = \overset{m}{\underset{i=1}{S}} R_i$ holds well by the Diens–Rescher implication function.

Theorem 1: Let *R* and R_i, respectively, be the relations for Rule 1 and subrule *i* of Rule 2 for $i = 1$ to n constructed by the Diens–Rescher implication function. Then, $R = \overset{m}{\underset{i=1}{S}} R_i(X_1, X_2, \ldots, X_n; Y_i)$.

Proof: Let $\mu_{A_i}(x_i)$ be the MF for x_i is A_i, for $i = 1$ to n. Then, by Diens–Rescher implication, we have

$$\begin{aligned}
&R(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) \\
&\quad = 1 - (\mu_{A_1}(x_1) t\ \mu_{A_2}(x_2) t \ldots t\ \mu_{A_n}(x_n)) \\
&s(\mu_{B_1}(y_1) s\ \mu_{B_2}(y_2) s \ldots s\ \mu_{B_m}(y_m)) \\
&\quad = \left(1 - \overset{n}{\underset{i=1}{t}} \mu_{A_i}(x_i)\right) s \left(\overset{m}{\underset{j=1}{s}} \mu_{B_j}(y_j)\right) \\
&\quad = \left\{\left(1 - \overset{n}{\underset{i=1}{t}} \mu_{A_i}(x_i)\right) s\ \mu_{B_1}(y_1)\right\} s \\
&\left\{\left(1 - \overset{n}{\underset{i=1}{t}} \mu_{A_i}(x_i)\right) s\ \mu_{B_2}(y_2)\right\} s \ldots \\
&s\left\{\left(1 - \overset{n}{\underset{i=1}{t}} \mu_{A_i}(x_i)\right) s\ \mu_{B_m}(y_m)\right\}.
\end{aligned}$$

Therefore, $R(x_1, \ldots, x_n; y_1, \ldots, y_m)$

$$\begin{aligned}
&\quad = R_1(x_1, x_2, \ldots, x_n; y_1) s \\
&R_2(x_1, x_2, \ldots, x_n; y_2) s \ldots s\ R_m(x_1, x_2, \ldots, x_n; y_m) \\
&\quad = \overset{m}{\underset{j=1}{s}} R_j(x_1, x_2, \ldots, x_n; y_j). \qquad (2)
\end{aligned}$$

Since (2) holds $\forall x_i \in X_i, i = 1$ to n, we obtain

$$R = \overset{m}{\underset{i=1}{S}} R_i(X_1, X_2, \ldots, X_n; Y_i)$$

thereby proving the statement of Theorem 1. □

Now, the inference obtained by Rule 1 is given by

$$T(\ldots T(T(A'_1, A'_2), A'_3), \ldots, A'_n) \circ R = \overset{n}{\underset{i=1}{T}} (A'_i) \circ R, \text{say}$$

where $T(A'_i, A'_j)$ is the t-norm between two vectors A'_i and A'_j, resulting in a matrix whose $(k \times l)$th component is the t-norm of $(\mu_{A'_i}(x_i^{(k)}), \mu_{A'_j}(x_j^{(l)}))$, and $\circ$ denotes the max-min composition operator, which acts like a typical matrix multiplication operator with the sum and the product being replaced by max and min operations, respectively. Here, $\mu_{A'_i}(x_i^{(k)})$ and $\mu_{A'_j}(x_j^{(l)})$ denote the kth and lth component of A'_i and A'_j, respectively. Theorem A1 in the Appendix shows that the result in Theorem 1 is also true for *Lukasiewicz* implication function.

Theorem 2 provides an interesting observation that decomposition of Rule 1 into the subrules of Rule 2 does not lose any information of Rule 1, as we can reconstruct the inference generated by Rule 1 by taking S-norm of the inferences obtained from Rule 2. Given $\overset{n}{\underset{i=1}{T}} (A'_i)$, and suppose that $R_1(X_1, X_2; Y_1)$ and $R_2(X_1, X_2; Y_2)$ are constructed with Lukasiewicz implication function, we by Theorem A2 given in the Appendix can prove the following statement:

$$\overset{n}{\underset{i=1}{T}} (A'_i) o (R_1 S R_2) = \left(\overset{n}{\underset{i=1}{T}} (A'_i) o R_1\right) S \left(\overset{n}{\underset{i=1}{T}} (A'_i) o R_2\right),$$

which can be also written as

$$\underset{\substack{y_1 \in Y_1 \\ y_2 \in Y_2}}{S} \{T(A'_i) o R_1, T(A'_i) o R_2\}.$$

1

The above result is used to prove Theorem 2. It is important to note that the above expression also holds, when R_1 and R_2 are constructed with Diens–Rescher implication.

Theorem 2: For Diens–Rescher- or Lukasiewicz-type implication, the inference obtained by Rule 1 is the OR of the inference obtained by the *j*th subrule of Rule 2 for *j* = 1 to *m*.

Proof: For Diens–Rescher- and Lukasiewicz-type implications, by (2), we have

$$R = \underset{\substack{y_1 \in Y_1 \\ y_2 \in Y_2 \\ \cdots \\ y_m \in Y_m}}{S} \{(R_1(x_1, x_2, \ldots, x_n; y_1), \ldots \quad (R_m(x_1, x_2, \ldots, x_n; y_m)\}.$$

Now, the inference obtained by Rule 1 is given by

$$\begin{aligned}
&\overset{n}{\underset{i=1}{T}} (A'_i) \circ R \\
&= \overset{n}{\underset{i=1}{T}} (A'_i) \circ \left\{ \underset{\substack{y_1 \in Y_1 \\ y_2 \in Y_2 \\ \cdots \\ y_m \in Y_m}}{S} (R_1(x_1, \ldots, x_n; y_1), \ldots, \quad (R_m(x_1, \ldots, x_n; y_m)) \right\} \\
&= \overset{n}{\underset{i=1}{T}} (A'_i) \circ \underset{\substack{y_1 y_2 \ldots y_{m-1} \in Y_1 \times Y_2 \times \cdots \times Y_{m-1} \\ y_m \in Y_m}}{S} (R_{123\ldots(m-1)}, R_m)
\end{aligned}$$

$$= \underset{\substack{y_1 y_2 \dots y_{m-1} \in Y_1 \times Y_2 \times \dots \times Y_{m-1} \\ y_m \in Y_m}}{S} \left\{ \overset{n}{\underset{i=1}{T}} (A'_i) \circ R_{123\dots(m-1)} \quad \overset{n}{\underset{i=1}{T}} (A'_i) \circ R_m \right\}$$

$$= \underset{\substack{y_1 y_2 \dots y_{m-2} \in Y_1 \times Y_2 \times \dots \times Y_{m-2} \\ y_{m-1} \in Y_{m-1} \\ y_m \in Y_m}}{S} \left\{ \overset{n}{\underset{i=1}{T}} (A'_i) \circ R_{12..(m-2)}, \quad \overset{n}{\underset{i=1}{T}} (A'_i) \circ R_{m-1}, \overset{n}{\underset{i=1}{T}} (A'_i) \circ R_m \right\}$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$= \underset{\substack{y_1 \in Y_1 \\ y_2 \in Y_2 \\ \dots\dots \\ y_m \in Y_m}}{S} \left\{ \overset{n}{\underset{i=1}{T}} (A'_i) \circ R_1, \overset{n}{\underset{i=1}{T}} (A'_i) \circ R_2, ..., \overset{n}{\underset{i=1}{T}} (A'_i) \circ R_m \right\}$$

$$= \underset{\substack{y_1 \in Y_1 \\ y_2 \in Y_2 \\ \dots\dots \\ y_m \in Y_m}}{S} \{B'_1, B'_2, \dots, B'_m\}$$

(by max-min compositional inference [56])

where $B'_j, j = 1, \dots, m$, is the inference made by the subrule j given $A'_i, j = 1, \dots, n$. This proves the theorem. □

B. Fuzzy Contraposition

In propositional logic, for any two propositions p and q, the tautology $p \rightarrow q \equiv \neg q \rightarrow \neg p$ is referred to as contraposition property [55]. To prove a propositional theorem, Wang [56] in his work proposed an extension of the contraposition property, which is formally introduced below.

Consider a propositional statement

$$p_1, p_2, p_3, \dots, p_n \rightarrow q_1, q_2, \dots, q_m \tag{3}$$

where $p_1, p_2, \dots, p_n$ and $q_1, q_2, \dots, q_m$ are atomic propositions, and comma in the left- and right-hand sides of the implication operator denotes AND and OR operators, respectively. The propositional statement (3) can equivalently be transformed to statement (4), shown below (for a formal proof, see [64]):

$$\neg q_1, \neg q_2, \dots, \neg q_m \rightarrow \neg p_1, \neg p_2, \neg p_3, \dots, \neg p_n. \tag{4}$$

This transformation of statement (3) to the statement (4) is called *extended contraposition property* and the latter statement is referred to as *extended contraposition rule*. Alternatively, the latter statement is hereafter referred to as the dual of the former (primal) statement. We now propose a fuzzy extension of the propositional contraposition property.

We verify that the extended contraposition property holds good for the Diens–Rescher implication function. Consider two fuzzy rules: Rule 1 introduced earlier and Rule 3 given below.

Rule 3: If y_1 is $\overline{B_1}$ and y_2 is $\overline{B_2}$ and $\dots y_m$ is $\overline{B_m}$
then x_1 is $\overline{A_1}$ or x_2 is $\overline{A_2}$ or $\dots$ or x_n is $\overline{A_n}$.

We would now prove in Theorem 3 that Rules 1 and 3 are equivalent.

Theorem 3: Rule 1 is equivalent to Rule 3, when the implication relations for these rules are constructed by the Diens–Rescher implication function.

Proof: Let $R_1(X_1, X_2, \dots, X_n; Y_1, Y_2, \dots, Y_m)$ and $R_3(Y_1, Y_2, \dots, Y_m; X_1, X_2, \dots, X_n)$ be the implication relations for Rules 1 and 3, respectively, constructed by the Diens–Rescher implication function. Then, with the defined t- and s-norm operators, we obtain the $(x_1, \dots, x_n, y_1, \dots, y_m)$th positional element of R_1 and $(y_1, \dots, y_m, x_1, \dots, x_n)$th positional element of R_3 and test their equality for all possible choices of $(x_1, \dots, x_n, y_1, \dots, y_m)$. Here

$$R_1(x_1, \dots, x_n, y_1, \dots, y_m) = \left(1 - \overset{n}{\underset{i=1}{t}} \mu_{A_i}(x_i)\right) s \left(\overset{m}{\underset{j=1}{s}} \mu_{B_j}(y_j)\right) \tag{5}$$

and

$$R_3(y_1, \dots, y_m, x_1, \dots, x_n) = \left(1 - \overset{m}{\underset{j=1}{t}} \mu_{\overline{B_j}}(y_j)\right) s \left(\overset{n}{\underset{i=1}{s}} \mu_{\overline{A_i}}(x_i)\right)$$

$$= \left(\overset{m}{\underset{j=1}{s}} \mu_{B_j}(y_j)\right) s \left(1 - \overset{n}{\underset{i=1}{t}} \mu_{A_i}(x_i)\right)$$

(by De Morgan s theorem)

$$= R_1(x_1, \dots, x_n, y_1, \dots, y_m) \quad \text{[by (5)]}.$$

The equality of $R_1(x_1, \dots, x_n, y_1, \dots, y_m)$ and $R_3(y_1, \dots, y_m, x_1, \dots, x_n)$ for all choices of $x_1, \dots, x_n, y_1, \dots, y_m$ ensures the equivalence of the two rules, thereby proving the theorem. □

Theorem A3 in the Appendix shows that the property of Theorem 3 also holds for the Lukasiewicz implication function. It can be easily proved using Theorems 2 and 3 that given Rule 1, we can get its dual form decomposing it into n subrules of the form "if y_1 is $\overline{B_1}$ and y_2 is $\overline{B_2}$ and $\dots$ y_m is $\overline{B_m}$, then x_i is $\overline{A_i}$" for $i = 1$ to n and derive the inference for x_i is $\overline{A_i}\ \forall i$. This is the basis of contraposition-based fuzzy abduction.

In the subsequent sections, for the sake of simplicity, we would use $\wedge$ and $\vee$ instead of t- and s-norms, unless we essentially require them.

III. Properties of Abductive Retrieval

In this section, we study a few interesting properties and conditions for "abductive retrieval" for two cases: first considering rules with single fuzzy proposition in antecedent and consequent, and later rules with multiple propositions in the consequent. While the former rules are considered for simplicity in analysis, the latter rules are taken up for the sake of completeness of the study. In the latter case, primal rules like Rule 1 with m fuzzy propositions in the consequent and n propositions in the antecedent are decomposed into n subrules each with one fuzzy proposition in consequent and m fuzzy propositions in the antecedent. This justifies the significance of primal rules with multiple propositions in the consequent.

A. Rules With Single Fuzzy Proposition in Antecedent and Consequent

Consider a fuzzy production rule "if x is A, then y is B," where x and y are two linguistic variables, and A and B are fuzzy sets defined on their respective universes U and V. By applying a fuzzy contraposition property, which is introduced in the previous section, we transform this primal rule into its equivalent dual form: if y is $\overline{B}$, then x is $\overline{A}$. Now, given the MF for y is $\overline{B'}$, we want to derive the MF of x is A'. Assuming that the MFs are available in vector form, i.e., $\overline{A} = [\overline{a_i}]_{1\times n}, \overline{B} = [\overline{b_j}]_{1\times m}$, and $\overline{B'} = [\overline{b'_j}]_{1\times m}$, by fuzzy modus ponens, we obtain

$$\overline{A'} = \overline{B'} \circ R \tag{6}$$

where R is the implication relation for the rule: If y is $\overline{B}$, then x is $\overline{A}$. Retrieval here means getting back $A' = A$, when $B' = B$ is supplied. Substituting $B' = B$ into (6), we obtain

$$\overline{A'} = \overline{B} \circ R. \tag{7}$$

A desirable property would be to obtain $A' = A$, but it is not possible for every implication function. Now, setting different implication functions for R, we determine the condition for $\overline{A'} = \overline{A}$, i.e., $(A' = A)$. Theorems 4 to 5 determine the conditions for $\overline{A'} = \overline{A}$ for two well-known implication functions: Diens–Rescher and Lukasiewicz.

Theorem 4: The Diens–Rescher implication retrieves $A' = A = [a_i]_{1\times n}$ from $B' = B = [b_j]_{1\times m}$ for the rule "if *x* is *A*, then *y* is *B*" when

$$\bigwedge_{i=1}^{n} a_i \geq b_k, \exists k \quad \text{and} \quad \bigwedge_{\substack{j=1\\ j\neq k}}^{m} \overline{b_j} \geq \bigvee_{i=1}^{n} a_i$$

jointly hold.

Proof: The dual of the given rule is "if y is $\overline{B}$, then x is $\overline{A}$." Therefore, by (7), we obtain $\overline{A'} = \overline{B} \circ R$, where R is the relational matrix of the dual rule constructed by the Diens–Rescher implication function. Replacing $\overline{A'} = [\overline{a'_i}], \overline{B} = [\overline{b_j}]$ and $R = [b_j \vee \overline{a_i}]$, in (7), we have

$$[\overline{a'_i}] = [\overline{b_j}] \circ [b_j \vee \overline{a_i}]$$

$$\text{or } \overline{a'_i} = \bigvee_{j=1}^{m} (\overline{b_j} \wedge (b_j \vee \overline{a_i}))$$

$$= \{\overline{b_k} \wedge (b_k \vee \overline{a_i})\} \vee \left\{ \bigvee_{\substack{j=1\\ j\neq k}}^{m} (\overline{b_j} \wedge (b_j \vee \overline{a_i})) \right\}$$

$$= \overline{b_k} \wedge (b_k \vee \overline{a_i}) = \overline{a_i}, \text{ when}$$

$$\overline{b_k} \wedge (b_k \vee \overline{a_i}) \geq \overline{b_j} \wedge (b_j \vee \overline{a_i}) \forall j, j \neq k \tag{8}$$

$\overline{a_i} \geq b_k$, and $b_k \vee \overline{a_i} \leq \overline{b_k}$ jointly hold. The latter two conditions together yield $\overline{a_i} \leq \overline{b_k}$, returning

$$b_k \leq a_i. \tag{9}$$

Since $a_i \geq b_k$ holds in (9) $\forall i$, we have

$$\bigwedge_{i=1}^{n} a_i \geq b_k, \quad \exists k \tag{10}$$

Substituting $\overline{a_i} \geq b_k$ and $b_k \vee \overline{a_i} \leq \overline{b_k}$ into (8), we have

$$\overline{a_i} \geq \overline{b_j} \wedge (b_j \vee \overline{a_i}) \quad \forall j, j \neq k$$

$$= (\overline{b_j} \wedge b_j) \vee (\overline{b_j} \wedge \overline{a_i}) \tag{11}$$

which holds if (12) and (13) jointly hold $\forall j, j \neq k$:

$$\overline{b_j} \leq \overline{a_i} \tag{12}$$

$$\overline{b_j} \geq \overline{b_j} \wedge b_j = \text{Min}(\overline{b_j}, b_j). \tag{13}$$

Now (13) holds if

$$\overline{b_j} \geq b_j \quad \forall j, j \neq k. \tag{14}$$

Combining (12) and (14), we have $b_j \leq \overline{b_j} \leq \overline{a_i}$,

$$\Rightarrow b_j \leq \overline{a_i}$$

$$\Rightarrow \overline{b_j} \geq a_i. \tag{15}$$

Since (15) holds for all $j, j \neq k$ and for all i, we have

$$\bigwedge_{\substack{j=1\\ j\neq k}}^{m} \overline{b_j} \geq \bigwedge_{i=1}^{n} a_i. \tag{16}$$

Thus, the retrieval is possible if (10) and (16) jointly hold. This completes the proof. □

Example 2: Let $A = [0.5\ \ 0.6\ \ 0.7]$ and $B = [0.2\ \ 0.1\ \ 0.25]$. Here, *m* = *n* = 3, and *k* = 2:

$$\bigwedge_{i=1}^{3} a_i = 0.5 > b_2 = 0.1, \text{ say.}$$

$$\bigwedge_{\substack{j=1\\ j\neq 2}}^{3} \overline{b_j} = 0.75 \geq \bigvee_{i=1}^{3} a_i = 0.7.$$

Therefore, the conditions stated in Theorem 4 are satisfied, and the theorem should hold. Let us verify it. We first construct the implication relation R using the Diens–Rescher function and then perform fuzzy modus ponens with $B' = B$ and obtain A', as shown in the following:

$$\overline{A'} = \overline{B'} \circ R$$

$$= [0.8 \quad 0.9 \quad 0.75] \circ \begin{bmatrix} 0.5 & 0.4 & 0.3 \\ 0.5 & 0.4 & 0.3 \\ 0.5 & 0.4 & 0.3 \end{bmatrix}$$

$$= [0.5 \quad 0.4 \quad 0.3]$$

$$\therefore A' = [0.5 \quad 0.6 \quad 0.7] = A.$$

The last result shows that A' has been correctly retrieved. We now derive the condition for correct retrieval by the Lukasiewicz implication function in Theorem 5. We first prove Lemma 1, which is required to prove the theorem.

Lemma 1: Let R_1 be a Lukasiewicz-type implication relation for the primal rule "if *x* is *A*, then *y* is *B*" and R_2 be the same for its dual rule "if *y* is $\overline{B}$, then *x* is $\overline{A}$." Then, $R_2(y, x) = R_1(x, y)$, for all *x*, *y*.

Proof: The Lukasiewicz implication relation for the primal rule is given by

$$R_1(x, y) = \text{Min}[1, (1 - \mu_A(x) + \mu_B(y))] \tag{17}$$

where $\mu_A(x)$ and $\mu_B(y)$ represent the MFs of x is A and y is B, respectively. The Lukasiewicz implication function for the dual rule is given by

$$R_2(y, x) = \text{Min}[1, (1 - (1 - \mu_B(y)) + (1 - \mu_A(x)))]$$

$$= \text{Min}[1, (1 - \mu_A(x) + \mu_B(y))] \tag{18}$$

$\therefore R_2(y, x) = R_1(x, y)$ holds for all x, y. □

Corollary 1: If R_1 and R_2 are in matrix form, then the primal and dual implication relations support $R_2 = R_1^T$, *where T denotes transposition.*

Proof: By Lemma 1, we obtain $R_2(y, x) = R_1(x, y)$. Since it holds for all $x, y, R_2 = R_1^T$ follows. □

Theorem 5: The Lukasiewicz implication retrieves $A' = A = [a_i]_{1\times n}$ from $B' = B = [b_j]_{1\times m}$ for the rule "if x is A, then y is B" if

$$b_k = 0, \exists k \quad \text{and} \quad \bigwedge_{i=1}^{n} \overline{a_i} \geq \bigvee_{\substack{j=1\\ j\neq k}}^{m} \overline{b_j}$$

jointly hold.

Proof: Let R be the Lukasiewicz-type implication relation for the rule "if x is A, then y is B." Then, by Lemma 1, we know that R^T would be the implication relation for the contraposition rule "if y is $\overline{B}$, then x is $\overline{A}$." Thus, by (6)

$$\overline{A'} = \overline{B'} \circ R^T$$
$$\text{or } [\overline{a'_i}] = [\overline{b'_j}] \circ [r_{ji}]$$
$$\text{or } \overline{a'_i} = \bigvee_{j=1}^{n} (\overline{b'_j} \wedge r_{ji}). \tag{19}$$

When $B' = B$, i.e., $\overline{b'_j} = \overline{b_j} \forall j$, (19) reduces to

$$\overline{a'_i} = \bigvee_{j=1}^{m} (\overline{b_j} \wedge (1 \wedge (1 - a_i + b_j))) \text{ (by definition of } r_{ji})$$
$$= \{\overline{b_k} \wedge (1 \wedge (1 - a_i + b_k)\} \vee \bigvee_{\substack{j=1\\ j\neq k}}^{m} (\overline{b_j} \wedge (1 \wedge (1 - a_i + b_j)))$$
$$= 1 - a_i, \text{ if the following conditions jointly hold:}$$

1) $\overline{b_k} \wedge (1 - a_i + b_k) \geq \bigvee_{\substack{j=1\\ j\neq k}}^{n} (\overline{b_j} \wedge (1 - a_i + b_j))$

2) $b_k = 0$.

Since $b_k = 0$, condition 1 yields

$$1 - a_i \geq \bigvee_{\substack{j=1\\ j\neq k}}^{n} (\overline{b_j} \wedge (1 - a_i + b_j)). \tag{20}$$

Since $(1 - a_i) \leq (1 - a_i + b_j)\forall j$, in order to satisfy (20), we need

$$\overline{b_j} < 1 - a_i + b_j \quad \forall j \neq k.$$

Substitution of $\overline{b_j} < 1 - a_i + b_j$ into (20) yields

$$1 - a_i \geq \bigvee_{\substack{j=1\\ j\neq k}}^{m} \overline{b_j}.$$

Since this has to hold for all i, we write

$$\bigwedge_{i=1}^{n} \overline{a_i} \geq \bigvee_{\substack{j=1\\ j\neq k}}^{m} \overline{b_j}.$$

Therefore, $\overline{a'_i} = 1 - a_i$, if $b_k = 0$, $\exists k$ and $\bigwedge_{i=1}^{n} \overline{a_i} \geq \bigvee_{\substack{j=1\\ j\neq k}}^{m} \overline{b_j}$ hold together. □

Definition 1: For any two vectors $B = [b_j]_{1\times m}$ and $B' = [b'_j]_{1\times m}$, we define $B' \leq B$, if $b'_j \leq b_j \forall j$.

Theorem 6: Consider a rule "if x is A, then y is B," where $A = [a_i]_{1\times n}$ and $B = [b_j]_{1\times m}$. Let the observed MF of the consequent be $B = [b'_j]_{1\times m}$ and the inferred MF for abduction be $A' = [a'_i]_{1\times n}$. Then, the necessary condition for $A' = A$ by Diens–Rescher-type implication-based reasoning with the contraposition rule "if y is $\overline{B}$, then x is $\overline{A}$," is given by $B \leq \overline{B'}$ and $\bigwedge_{i=1}^{n} a_i \geq \bigvee_{j=1}^{m} b'_j$.

Proof: Proceeding like the proof of Theorem 5, we arrive at

$$\overline{a'_i} = \bigvee_{j=1}^{m} (\overline{b'_j} \wedge (b_j \vee \overline{a_i}))$$
$$\therefore \overline{a'_i} = \overline{a_i}$$
$$\text{if } (\overline{a_i} \vee b_j) \leq \overline{b'_j} \quad \forall j = 1 \text{ to } m \tag{21}$$

and

$$\overline{a_i} \geq b_j \quad \forall j = 1 \text{ to } m. \tag{22}$$

Substituting (22) into (21), we have

$$\overline{a_i} \leq \overline{b'_j} \quad \forall j = 1 \text{ to } m. \tag{23}$$

From (22) and (23), we obtain

$$b_j \leq \overline{a_i} \leq \overline{b'_j} \quad \forall j. \tag{24}$$

Since $b_j \leq \overline{b'_j} \quad \forall j$, we obtain

$$B \leq \overline{B'}. \tag{25}$$

Since (23) holds for all j, we have

$$\overline{a_i} \leq \bigwedge_{j=1}^{m} \overline{b'_j}. \tag{26}$$

Again, as (26) holds for all i, we obtain

$$\bigvee_{i=1}^{n} \overline{a_i} \leq \bigwedge_{j=1}^{m} \overline{b'_j}.$$

Taking complement on both sides and simplifying by De Morgan's law, we find

$$\bigwedge_{i=1}^{n} a_i \geq \bigvee_{j=1}^{m} b'_j$$

i.e., given B', we can obtain $A' = A$ if $B \leq \overline{B'}$ and $\bigwedge_{i=1}^{n} a_i \geq \bigvee_{j=1}^{m} b'_j$ jointly hold.

Theorem 7: Given $B' = [b'_1 \ b'_2 \ldots b'_m]$ and $C' = [c'_1 \ c'_2 \ldots c'_m]$ be two MFs, such that $b'_i = c'_i + \delta c_i, \delta c_i \geq 0, \forall i = 1, \ldots, m$. Let A'_1 and A'_2 be the inferred MFs corresponding to measured MFs for B' and C', respectively, obtained by extended contraposition rule: if y is $\overline{B}$, then x is $\overline{A}$. If $B' \geq C'$, then $A'_1 \geq A'_2$.

Proof: Since A'_1 is the inferred MF for the given MF B', by (6), we have

$$\overline{A'_1} = \overline{B'} \circ R \tag{27}$$

where R is the relational matrix for the given fuzzy contraposition rule.

Let $\Delta c = [\delta c_1 \ \ \delta c_2 \ \ldots \ \delta c_m$, where $\delta c_1 \geq 0 \ \forall_i$. Now substituting $B' = C' + \Delta C$ in (27), we have

$$\overline{A'_1} = \overline{C' + \Delta C} \circ R$$
$$= [(\overline{c'_1 + \delta c'_1})(\overline{c'_2 + \delta c'_2}) \ldots (\overline{c'_m + \delta c'_m})] \circ R$$
$$= [\{1 - (c'_1 + \delta c'_1)\} \ldots \{1 - (c'_m + \delta c'_m)\}] \circ [r_{ij}]_{m\times n}$$

$$= \Big[\Big\{\overset{m}{\underset{i=1}{\vee}}(1-(c_i'+\delta c_i'))\wedge r_{i1}\Big\}\ldots\Big\{\overset{m}{\underset{i=1}{\vee}}(1-(c_i'+\delta c_i')\wedge r_{in}\Big\}\Big]$$

$$\leq \Big[\Big\{\overset{m}{\underset{i=1}{\vee}}(1-c_i')\wedge r_{i1}\Big\}\Big\{\overset{m}{\underset{i=1}{\vee}}(1-c_i')\wedge r_{i2}\Big\}$$

$$\ldots\Big\{\overset{m}{\underset{i=1}{\vee}}(1-c_i')\wedge r_{in}\Big\}\Big]\Big(\because \Big\{\overset{m}{\underset{i=1}{\vee}}(1-c_i')\wedge r_{ik}\Big\}$$

$$\geq \Big\{\overset{m}{\underset{i=1}{\vee}}(1-(c_i'+\delta c_i')\wedge r_{ik}\Big\}, \forall k\Big)$$

$$= \overline{C'}\circ R = \overline{A_2'} \text{ (by definition)}$$

or $\overline{A_1'} \leq \overline{A_2'}$

$\therefore\ A_1' \geq A_2'$.

Corollary: Suppose A_i' is the inferred MF corresponding to the measured MF B_i', for $i = 1$ to p. Given that $B_1' \geq B_2' \geq \cdots \geq B_p'$, then $A_1' \geq A_2' \geq \cdots \geq A_p'$.

Theorem 8: The contraposition-based fuzzy abduction with the rule "if x is A, then y is B" using Diens–Rescher implication yields $a_i' \geq a_i$ if $\overset{m}{\underset{j=1}{\wedge}} b_j' \geq \overset{n}{\underset{i=1}{\vee}} a_i$, and $a_i' \leq a_i$ if $\overset{n}{\underset{i=1}{\wedge}} a_i \geq \overset{m}{\underset{j=1}{\wedge}} b_j$, where $A' = [a_1'\ a_2' \ldots a_n']$ is the inferred MF corresponding to the measured MF $B' = [b_1'\ b_2' \ldots b_m']$.

Proof: Proceeding like the Proof of Theorem 5, we obtain

$$\overline{a_i'} = \overset{m}{\underset{j=1}{\vee}}(\overline{b_j'}\wedge(b_j\vee\overline{a_i}))$$

$$= \overset{m}{\underset{j=1}{\vee}}\{(\overline{b_j'}\wedge b_j)\vee(\overline{b_j'}\wedge\overline{a_i})\}. \quad (28)$$

Now, if $\overline{b_j'}\wedge\overline{a_i} \geq \overline{b_j'}\wedge b_j$ then by (28), we have

$$\overline{a_i'} = \overset{m}{\underset{j=1}{\vee}}(\overline{b_j'}\wedge\overline{a_i}) = \Big(\overset{m}{\underset{j=1}{\vee}}\overline{b_j'}\Big)\wedge\overline{a_i} \quad (29)$$

$$\text{or } \overline{a_i'} = \overset{m}{\underset{j=1}{\vee}}\overline{b_j'} \quad (30)$$

$$\text{if } \overset{m}{\underset{j=1}{\vee}}\overline{b_j'} \leq \overline{a_i} \quad (31)$$

or $\overline{a_i'} \leq \overline{a_i}$ [combining (30) and (31)] or $a_i' \geq a_i$, if $\overset{m}{\underset{j=1}{\vee}}\overline{b_j'} \leq \overline{a_i}$.

Since (31) holds $\forall i$, we have

$$\overset{m}{\underset{j=1}{\vee}}\overline{b_j'} \leq \overset{n}{\underset{i=1}{\wedge}}\overline{a_i}$$

or $\overset{m}{\underset{j=1}{\wedge}} b_j' \geq \overset{n}{\underset{i=1}{\vee}} a_i$ (by De Morgan's theorem).

$$\text{Therefore, } a_i' \geq a_i \text{ if } \overset{m}{\underset{j=1}{\wedge}} b_j' \geq \overset{n}{\underset{i=1}{\vee}} a_i. \quad (32)$$

$$\text{Again, in (28), if } \overline{b_j'}\wedge\overline{a_i} \leq \overline{b_j'}\wedge b_j \quad (33)$$

$$\overline{a_i'} = \overset{m}{\underset{j=1}{\vee}}(\overline{b_j'}\wedge b_j)$$

$$\geq \overset{m}{\underset{j=1}{\vee}}(\overline{b_j'}\wedge\overline{a_i}) \text{ [by (33)]}$$

$$\text{or } a_i' \leq \overline{\overset{m}{\underset{j=1}{\vee}}(\overline{b_j'}\wedge\overline{a_i})} = \Big(\overset{m}{\underset{j=1}{\wedge}} b_j'\Big)\vee a_i$$

$$\text{or } a_i' \leq a_i \text{ if } a_i \geq \overset{m}{\underset{j=1}{\wedge}} b_j'. \quad (34)$$

$$\text{Since } a_i \geq \overset{m}{\underset{j=1}{\wedge}} b_j'\ \forall i, \text{ we have } \overset{n}{\underset{i=1}{\wedge}} a_i \geq \overset{m}{\underset{j=1}{\wedge}} b_j'. \quad (35)$$

Thus, $a_i' \leq a_i$ if $\overset{n}{\underset{i=1}{\wedge}} a_i \geq \overset{m}{\underset{j=1}{\wedge}} b_j'$

The theorem follows from (32) and (35).

1) Fuzzy Abduction by Generalized Modus Tollens: The analysis of fuzzy abduction that we have done so far deals with the extended contraposition property. However, abduction can also be performed using GMT. Consider the rule "if x is A, then y is B." Suppose we are given the MF of y is B' and that we want to obtain the MF of x is A'. Now, by GMT, we have

$$A' = B'\circ R^T \quad (36)$$

where $A' = [a_i']_{1\times n}$, $B' = [b_j']_{1\times m}$ are the MFs of x is A' and y is B', respectively, and $R = [\overline{a_i}\vee b_j]_{n\times m}$ is the implication relation for the given rule by Diens–Rescher implication with $A = [a_i]_{1\times n}$ and $B = [b_j]_{1\times m}$. Substituting $A' = [a_i']$, $B' = [b_j']$, and $R = [\overline{a_i}\vee b_j]_{n\times m}$ into (36), we have

$$[a_i'] = [b_j']\circ[b_j\vee\overline{a_i}]. \quad (37)$$

Theorem 9 below determines the necessary conditions for abductive retrieval by GMT.

Theorem 9: The necessary conditions for $A' = \overline{A}$ by GMT for the rule "if x is A, then y is B" using the Diens–Rescher implication function are $B \leq B'$ and $\overset{n}{\underset{i=1}{\wedge}}\overline{a_i} \geq \overset{m}{\underset{j=1}{\vee}} b_j$, where B' is the observed MF.

Proof: Expanding max-min composition in (37) in pointwise notation, we obtain

$$a_i' = \underset{\forall j}{\vee}(b_j'\wedge(b_j\vee\overline{a_i}))$$

$$= \underset{\forall j}{\vee}\{(b_j'\wedge b_j)\vee(b_j'\wedge\overline{a_i})\}$$

$$= \overline{a_i}$$

if the following conditions jointly hold:

$$(b_j'\wedge\overline{a_i}) \geq (b_j'\wedge b_j) \quad \forall j \quad (38)$$

$$\text{and } \overline{a_i} \leq b_j' \quad \forall j. \quad (39)$$

From (38), we obtain

$$\overline{a_i} \geq b_j. \quad (40)$$

Thus, from (39) and (40), we obtain

$$b_j \leq \overline{a_i} \leq b_j' \quad \forall j \quad (41)$$

i.e., in vector form, $B \leq B'$. Further, as (41) holds for all i, j, we have

$$\overset{n}{\underset{i=1}{\wedge}}\overline{a_i} \geq \overset{m}{\underset{j=1}{\vee}} b_j. \quad (42)$$

Therefore, the necessary conditions for abductive retrieval are $B \leq B'$ and $\overset{n}{\underset{i=1}{\wedge}}\overline{a_i} \geq \overset{m}{\underset{j=1}{\vee}} b_j$. □

Note that the necessary condition for abductive retrieval for GMT-based reasoning differs significantly from the contraposition-based reasoning as evident from Theorems 6 and 9.

We now prove one very interesting property of GMT-based abductive reasoning in the following theorem.

Theorem 10: The GMT-based abductive reasoning with the rule "if x is A, then y is B" using Diens–Rescher implication returns

$$A' \le \overline{A}, \text{ if } \underset{\forall i}{\wedge} \overline{a_i} \ge \underset{\forall j}{\vee} b_j$$

$$\text{and } A' \ge \overline{A} \text{ if } \underset{\forall i}{\vee} \overline{a_i} \le \underset{\forall j}{\vee} b'_j \text{ and } \underset{\forall j}{\wedge} b_j \ge \underset{\forall i}{\vee} \overline{a_i}$$

where $A = [a_i]$ and $B = [b_j]$ are measured MFs, and $B' = [b'_j]$ and $A' = [a'_i]$ are observed and inferred MFs.

Proof: Proceedings like the proof of Theorem 9, we obtain

$$\begin{aligned} a'_i &= \underset{\forall j}{\vee}\{(b'_j \wedge b_j) \vee (b'_j \wedge \overline{a_i})\} \\ &= \underset{\forall j}{\vee}(b'_j \wedge \overline{a_i}) \\ &\quad \text{if } (b'_j \wedge \overline{a_i}) \ge (b'_j \wedge b_j) \\ &\quad \text{or } \overline{a_i} \ge b_j \qquad (43) \\ \text{or } a'_i &= (\underset{\forall j}{\vee} b'_j) \wedge \overline{a_i} \\ \text{or } a'_i &\le \overline{a_i} \text{ if } \overline{a_i} \ge b_j \forall i, j \\ \text{or } a'_i &\le \overline{a_i} \\ \text{if } &\underset{\forall i}{\wedge} \overline{a_i} \ge \underset{\forall j}{\vee} b_j. \qquad (44) \end{aligned}$$

Again from (43), we obtain

$$\begin{aligned} a'_i &= \underset{\forall j}{\vee}(b'_j \wedge b_j) \\ &\text{if } \overline{a_i} \le b_j \\ &\ge \underset{\forall j}{\vee}(b'_j \wedge \overline{a_i}) (\because \overline{a_i} \le b_j) \\ &= (\underset{\forall j}{\vee} b'_j) \wedge (\overline{a_i}) \\ &= \overline{a_i} \qquad (45) \\ &\text{if } \overline{a_i} \le \underset{\forall j}{\vee} b'_j. \qquad (46) \end{aligned}$$

Therefore, $a'_i \ge \overline{a_i}$, if $\overline{a_i} \le b_j$ and $\overline{a_i} \le \underset{\forall j}{\vee} b'_j$ hold jointly $\forall i, j$ or $a'_i \ge \overline{a_i}$ if

$$\underset{\forall i}{\vee} \overline{a_i} \le \underset{\forall j}{\wedge} b_j \text{ and } \underset{\forall i}{\vee} \overline{a_i} \le \underset{\forall j}{\vee} b'_j. \qquad (47)$$

The theorem follows from (44) and (47). □

Example 3: Let $A = [0.1\ \ 0.2\ \ 0.3]$ and $B = [0.4\ \ 0.5\ \ 0.6]$; we construct $R = [r_{ij}] = [b_j \vee \overline{a_i}]$. Now, suppose $B' = [0.5\ \ 0.6\ \ 0.7]$ so that $\underset{\forall i}{\wedge} \overline{a_i} = 0.7 \ge \underset{\forall j}{\vee} b_j = 0.6$. Therefore, by Theorem 10, $A' \le \overline{A}$. We test it by computing $A' = B' \circ R = [0.7\ 0.7\ 0.7] \le \overline{A}$. Therefore, the condition for $A' \le \overline{A}$ is tested.

Now to test the condition for $A' \ge \overline{A}$, consider $A = [0.1\ 0.2\ 0.3]$, $B = [0.9\ 0.94\ 0.96]$, and $B' = [0.9\ 0.92\ 0.95]$. It is easy to see that $\underset{\forall i}{\vee} \overline{a_i} \le \underset{\forall j}{\vee} b'_j$ and $\underset{\forall i}{\vee} \overline{a_i} \le \underset{\forall j}{\wedge} b_j$ are satisfied. Now constructing R with the new B and old A, we compute that $A' = B' \circ R = [0.9\ 0.94\ 0.95] \ge \overline{A}$ is satisfied.

B. Rules With Multiple Fuzzy Propositions in Consequent

Here, we determine the conditions for abductive retrieval with rules having multiple fuzzy propositions in consequent for Lukasiewicz- and Diens–Rescher-based implication functions as indicated by Theorems 11 and 12, respectively.

Theorem 11: Let $A = [a_i]_{1\times n}$, $B = [b_j]_{1\times m}$ and $C = [c_k]_{1\times p}$. The contraposition-based fuzzy abduction with the primal rule "if x is A, then y is B or z is C" employing Lukasiewicz-type implication function retrieves $A' = A$ if the following conditions jointly hold:

1) $b_u = c_v = 0, \quad \text{for } \exists u, v.$
2) $\overline{a_i} \ge \overline{b_j} \wedge \overline{c_k} \quad \forall j, k, j \ne u, k \ne v.$

Proof: See the Appendix.

Theorem 12: The contraposition-based fuzzy abduction with the primal rule "if x is A, then y is B or z is C" employing the Diens–Rescher-type implication function retrieves $A' = A$, if the following conditions jointly hold:

1) $b_u = 0, c_v = 0, \quad \text{for any } u, v.$
2) $(\overline{b_j} \wedge \overline{c_k}) \le \overline{a_i} \quad \forall j, k, j \ne u, k \ne v.$

Proof: See the Appendix.

IV. Abduction With Multiple Chained Rules

Chaining is an important issue in fuzzy reasoning. Consider two rules R_i and R_j, where the consequent part of rule R_i includes y_1 is B_1 and the antecedent part of rule R_j includes y_1 is B'_1, where y_1 is a fuzzy linguistic variable, and B_1 and B'_1 are two fuzzy sets on the same universe. We call these two rules to be *chained* (interdependent). The above definition of chaining is explicitly visualized below with the following two rules:

Rule R_i: If x_1 is A_1 and x_2 is A_2
then y_1 is B_1 or y_2 is B_2
Rule R_j: If y_1 is B'_1 and z_1 is C_1
then w_1 is D_1 or w_2 is D_2

where $x_1, x_2, y_1, y_2, z_1, w_1$, and w_2 are linguistic variables, and $A_1, A_2, B_1, B_2, B'_1, C_1, D_1$, and D_2 are fuzzy sets (linguistic values) on respective universes.

Petri like nets [57] are probably one of the efficient tools to represent chaining. There exists an extensive literature [58]–[60], [65] on fuzzy reasoning using Petri nets. Researchers prefer Petri like nets in reasoning particularly for their 1) structural advantage in representing knowledge [54] and 2) power of automated reasoning, by which they can automatically identify the "enabled" rules for firing. Here, we would develop a new functional property of Petri nets utilizing its structural properties [61] to design an algorithm for contraposition-based fuzzy abduction. Because of the introduction of the new functional property, the proposed model of computation has operational difference with classical Petri net models [63], and therefore, it is referred to as FLN. The proposed algorithm for fuzzy abduction realized with FLN can perform reasoning even in presence of "complex chaining of rules," where firing of a third rule depends on the results of joint firing of two (or more) rules. Such complexity in abduction can be handled by imposition of stringent control on rule firing and resource sharing among rules (for example, inferences/resources generated by a rule may be used for firing of other rules). The proposed abductive reasoning algorithm takes care of all the necessary control actions required to perform abduction with FLN by utilizing its structural and functional properties, and thereby relieves the users from the additional burden of determining the firing sequence of rules and resource sharing.

Here, we propose a primal and dual model of FLN, where the primal net is built up with typical fuzzy IF–THEN rules, which are also called primal rules, whereas the dual net is constructed with a set of rules, obtained by transformation of the rules used in the primal net. The transformation in the present context is fuzzy extension of the contraposition property. After we transform a primal net to its dual form, abduction can be performed on the dual net by the reasoning mechanism introduced earlier (in Section II). The FLN model we shall use for abduction with classical fuzzy sets is described next. An FLN can be defined as a six-tuple:

$$\text{FLN} = \{P, Tr, D, I, O, R\}$$

where we have the following.

1) $P = \{p_1, p_2, \ldots, p_N\}$ is a finite set of places.
2) $Tr = \{tr_1, tr_2, \ldots, tr_M\}$ is a finite set of transitions.
3) $D = \{d_1, d_2, \ldots, d_N\}$ is a finite set of fuzzy propositions, where each d_i is associated with a place p_i.
4) $I : Tr \to P$ is the input function, representing a mapping from transitions to their input places.
5) $O : Tr \to P$ is the output function, representing a mapping from transitions to their output places.
6) $n_i : d_i \to [0, 1]^l$ is an association function, representing a mapping from fuzzy proposition $d_i = x_i$ is A_i to an l-dimensional MF $\mu_{A_i}(x_i)$ describing memberships in [0,1] for l distinct values of x_i, while n_i denotes that measured MF for x_i is A_i, and n'_i denotes observed MF for the position: x_i is A'_i.
7) $R_{ij} : tr_i \times p_j \to [0, 1] \times [0, 1]$ represents a fuzzy relational matrix R_{ij} at an arc connected between transition tr_i to place p_j, where $p_j \in \text{O}(tr_i)$.
8) $C \subseteq P$ is a set of axioms, where for any $p_i \in C, p_i \notin O(tr_j), \exists j,$ where $tr_j \in Tr$.
9) $T \subseteq P$ is a set of terminal places, where for any $p_i \in T, p_i \notin I(tr_j), \exists j,$ where $tr_j \in Tr$.

Example 4: In Fig. 1, $P = \{p_1, p_2, \ldots, p_7\}$, Tr $= \{tr_1, tr_2\}$, and $D = \{d_1, d_2, \ldots, d_7\}$, where $d_1 = x_1$ is A_1, $d_2 = x_2$ is A_2, $d_3 = y_1$ is B_1, $d_4 = z_1$ is C_1, $d_5 = y_2$ is B_2, $d_6 = w_1$ is D_1, and $d_7 = w_2$ is D_2.

$$I(tr_1) = \{p_1, p_2\}, \quad I(tr_2) = \{p_3, p_4\}, \quad O(tr_1) = \{p_3, p_5\}$$

$$O(tr_2) = \{p_6, p_7\}. n_1 = \mu_{A_1}(x_1), \quad n_2 = \mu_{A_2}(x_2)$$

$$n_3 = \mu_{B_1}(y_1), \quad n_4 = \mu_{C_1}(z_1), \quad n_5 = \mu_{B_2}(y_2)$$

$$n_6 = \mu_{D_1}(w_1), \text{ and } n_7 = \mu_{D_2}(w_2).$$

R_{13} and R_{15} are relational matrices representing implication relations associated with the arcs: $tr_1 \times p_3$ and $tr_1 \times p_5$, respectively. Similarly, R_{26} and R_{27} are the relational matrices associated with the arcs $tr_2 \times p_6$ and $tr_2 \times p_7$, respectively.

1) Let $d_j \forall j$ be a fuzzy proposition of the form: x is A. Then, for each rule R_i of the format "*if* d_1 *and* d_2 *and* $\ldots$ d_k, *then* d_{k+1} *or* d_{k+2} *or*$\ldots$ *or* d_{k+n}," we create a transition tr_i with input places $p_1, p_2, \ldots, p_k$ and output places $p_{k+1}, p_{k+2}, \ldots, p_{k+n}$, where d_j is associated with $p_j \forall j$. Assign MF n_j to $d_j \exists j$, wherever available.
2) Let $d_j = x$ is A and $d'_j = x$ is A', where d_j occurs as the consequent of rule R_i and d'_j occurs as the antecedent of rule R_k; then, we define place $p_j \varepsilon \{0(tr_i) \cap I(tr_k)\}$ for two transitions tr_i and tr_k corresponding to rules R_i and R_k, respectively, and associate $d'_j = x$ is A' to place p_j.
3) Steps 1 and 2 should be repeatedly used until all the given rules are encoded into FLN.

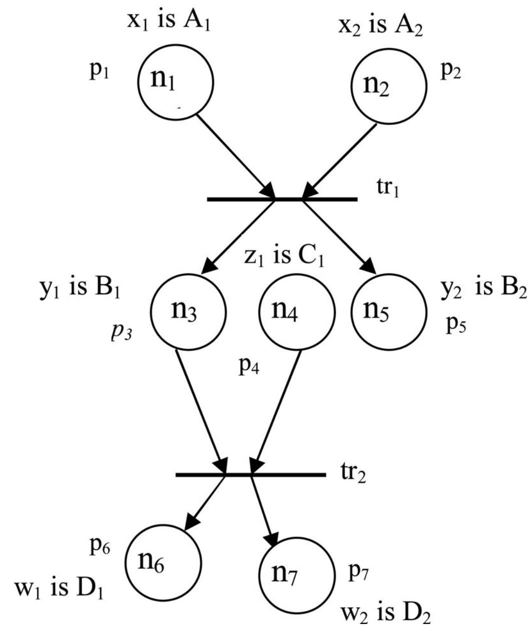

Fig. 1. FLN used to illustrate its parameters in Example 6.

Theorem 13 below provides an interesting basis to construct a dual FLN from its primal counterpart.

Theorem 13: If U is a primal FLN constructed with a set of primal rules $R = \{R_1, R_2, \ldots, R_n\}$, then the dual FLN V can be constructed by complementing the fuzzy propositions d_i (i.e., MF of d_i) at place p_i for all i and reversing all the arrows in the primal FLN.

Proof: We are given a primal FLN U containing encoded rules $R_1, R_2, \ldots, R_n$. Let R_i be the ith rule of the form "*if* d_1 *and* d_2 *and* $\ldots$ *and* d_k *then* d_{k+1} *or* d_{k+2} *or* $\ldots$ *or* d_{k+m}" encoded in FLN U using places $p_1, p_2, \ldots, p_k$; $p_{k+1}, p_{k+2}, \ldots, p_{k+m}$ and a transition tr_i such that $p_j \ \varepsilon \ I\ (tr_i)\ \forall j = 1, 2, \ldots, k$ and $p_l \ \varepsilon \ O(tr_i)\ \forall l = k + 1,\ k + 2, \ldots, k + m$. When R_i is transformed into its dual form using the fuzzy contraposition property, we call it R'_i, where R'_i is given by "*if not* d_{k+1} *and not* d_{k+2} *and* $\ldots$ *not* d_{k+m}, *then not* d_1 *or not* d_2 *or* $\ldots$ *or not* d_k." If R'_i is encoded into place-transition representation, then $p_l \ \varepsilon \ I\ (tr'_i)$, where $l = k + 1, k + 2, \ldots, k + m$ and $p_j \ \varepsilon \ O\ (tr'_i)$, $j = 1, 2, \ldots, k$, where d_i is not associated with p_i and d_l is not associated with p_l. In other words, tr'_i and its input and output places can be constructed from tr_i and its input and output places by reversing arrows and complementing all propositions. Thus, the theorem holds good for independent rules of the primal net.

Now, we show that the theorem also holds good for chained rules. Let R_i and R_j be chained rules in U, then there must be at least one place p_s, such that $p_s \ \varepsilon \ O(tr_i)$ and $p_s \ \varepsilon \ I(tr_j)$, where tr_i and tr_j are the transitions corresponding to rules R_i and R_j, respectively. After transformation of rules R_i and R_j by the fuzzy contraposition property, we obtain new transitions tr'_i and tr'_j corresponding to tr_i and tr_j, respectively, such that $p_s \ \varepsilon \ O(tr'_j)$ and $p_s \ \varepsilon \ I \ (tr'_i)$, which means reversal of arrows in the dual net V with respect to the primal net around p_s.

Further, because of transformation of the rules tr_i and tr_j using the fuzzy contraposition property, the propositions in the input/output places of tr'_i and tr'_j would be all complemented with respect to those in the primal net around tr_i and tr_j. Thus, the proposed theorem is satisfied for chained rules as well.

Now, if we order the rules in set R in a manner that two successive rules of the set are chained (dependent), then application of the fuzzy contraposition property to these rules results in chained rules with reverse-ordered and negated propositions. The reverse-ordered chained rules thus obtained have a corresponding FLN structure, called the dual net, which can be obtained by reversing arrows of the transitions and negating all propositions in the primal net U. Hence, the theorem follows. □

A. Enabling and Firing Condition of Transitions

A transition tr_i is enabled, if for all $j, p_j \in I(tr_i)$ possess fuzzy MFs n_j. An enabled transition fires and new MFs are generated at the arc $tr_i \times p_s$, where $p_s \in O(tr_i)$.

Let $R_{i,s}$ be the relational matrix associated with the arc: $tr_i \times p_s$. Then, the fuzzy truth token (FTT) at the arc $tr_i \times p_s$ is obtained as $(\underset{\forall j}{T}(n'_j))^T \circ R_{i,s}$, where n'_j is the observed MF of the fuzzy proposition d_j at place p_j. If $p_s \in \overset{k}{\underset{i=1}{\cap}} O(tr_i)$ and all tr_i for $i = 1$ to k, for any integer $k > 1$, are fired, then the fuzzy MF at place p_s is obtained as $n'_s = \overset{n}{\underset{i=1}{\vee}} \{(\underset{\forall j}{T}(n'_j))^T \circ R_{i,s}\}$, where $\vee$ denotes componentwise ORing of the results $(\underset{\forall j}{T}(n'_j))^T \circ R_{i,s}$ for each i.

The relational matrix $R_{i,s}$ evaluated using the Diens–Rescher implication function is given by $\{\overline{(\underset{\forall j}{T}(n_j))^T}\}\Phi(n_s)$, where Φ denotes Min-Max composition operator, which is computed in the same manner as done for Max-Min composition, by swapping the Min and Max operators.

For example, in Fig. 2, we have $R_{1,s} = (\overline{t(n_1, n_2)})^T \, \Phi(n_2)$, $R_{2,s} = (\overline{t(n_3)})^T \, \Phi(n_s)$ and $n'_s = \{(t(n'_1, n'_2))^T \circ R_{1,s}\} \vee \{(t(n'_3))^T \circ R_{2,s}\}$.

It may be noted that like classical Petri nets [61], tokens (here MFs) are not removed from the input places of a fired transition. The principles of enabling and firing of transition, as introduced above for the primal net, are also applicable for the dual net. An algorithm for abduction is presented here based on the enabling and firing conditions of transitions in a dual net.

The proposed algorithm requires the measured MFs at all places of the primal net and observed MFs at the terminal places $p_i \ \forall i$, where $p_i \notin I(tr_j)$, $\exists j$ (i.e. places which are not input places of any transition). The algorithm transforms a given primal net to its dual form by reversing arrows and complementing measured MFs at all places to construct relational matrices associated with the arc: $tr_j \times p_s \ \forall j$, where $p_s \ \varepsilon \ O(tr_j)$. The terminal places of the primal net now becomes axioms ($p_i \ \forall i$, where $p_i \notin O(tr_k)$, $\exists k$) of the dual net, and observed MFs at the terminal places in the primal net are now complemented. This is the initialization part of the algorithm.

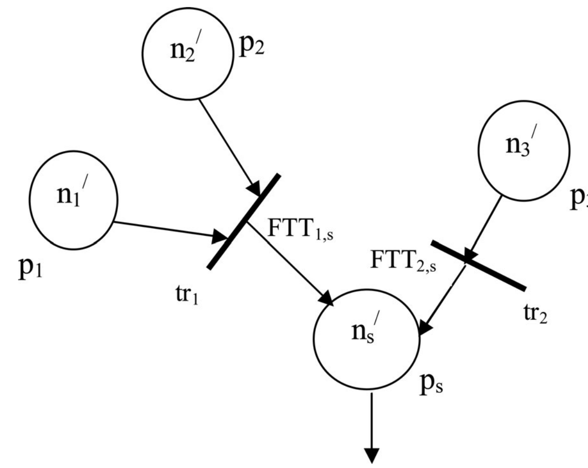

Fig. 2. Computation of MF n'_s at a place p_s from the measured MFs n'_1, n'_2, and n'_3 of respective places p_1, p_2, and p_3: $n'_s = \mathrm{FTT}_{1,s} \vee \mathrm{FTT}_{2,s}$, where $\mathrm{FTT}_{1,s} = (t(n'_1, n'_2))^T \circ R_{1,s}$, and FTT2,$s = (n'_3)^T \circ R_{2,s}$.

The rest of the algorithm checks the enabling condition of transitions in the dual net and computes FTT at $tr_j \times p_s$, where $p_s \ \varepsilon \ O(tr_j)$, and MF $\overline{n'_s}$ at place p_s. If p_s is an output place of two or more transitions $tr_j, \exists j$, then until $\mathrm{FTT}_{j,s} \ \forall j$ are computed, we cannot compute $\overline{n'_s}$. The algorithm is terminated when the computed MFs at the terminal places of the dual net are obtained. We complement the resulting $\overline{n'_s}$ at the terminal place $p_s \ \forall s$ of the dual net and thus obtain n_s

B. Pseudocode of the Fuzzy Abduction Algorithm

1) Transform a primal FLN into its dual form by reversing arrows and complementing measured MF n_i at each place p_i of the primal net. In addition, complement the observed MF n'_i at each terminal place p_i of the primal FLN and map it at place p_i of the dual net. Initialize set of current places C = set of axioms in the dual net (i.e., set of terminal places in the primal net) and T = set of terminal places in the dual net (i.e., set of axioms in the primal net).
2) For each place $p_i \in C$
   a) If $p_i \in I(tr_j)\exists j$
      Then, if each input place p_k of tr_j possesses MF, then evaluate FTT at the arc $tr_j \times p_s, \exists s$ where $p_s \in O(tr_j)$, as $\mathrm{FTT}_{j,s} = (\underset{\forall k}{T}(\overline{n'_k}))^T \circ R_{j,s}$ and $R_{j,s} = (\overline{\underset{\forall k}{T}(\overline{n_k})})^T \, \Phi\overline{n_s}$, where Φ is Min-Max composition operator.
      End if;
   b) If $p_s \in \underset{\forall l}{\cap} O(tr_l)$ and $\mathrm{FTT}_{l,s} \forall l$ are known,
      then $\overline{n_s} \leftarrow \underset{\forall l}{\vee} \mathrm{FTT}_{l,s}$;
      If p_s is the single output place of tr_l, then $\overline{n_s} \leftarrow \mathrm{FTT}_{j,s}$. If computation of $\overline{n_s}$ is performed, then $C \leftarrow C \cup \{p_s\}$;

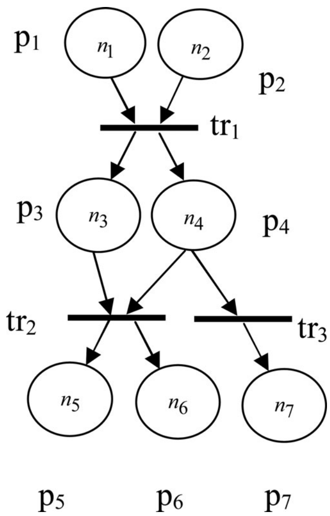

Fig. 3. Primal FLN used to illustrate the abductive reasoning algorithm.

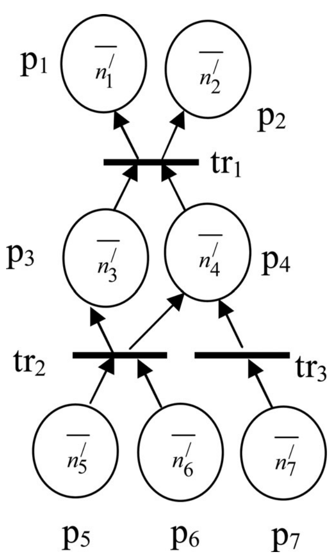

Fig. 4. Dual net equivalent to the Primal FLN of Fig. 3.

c) If $p_i \in \bigcap_{r=1}^{m} I(tr_r)$ then if $\mathrm{FTT}_{r,j}$ where $p_j \in O(tr_r)$, $\forall j, \forall r$ are known then $C \leftarrow C - \{p_i\}$.
For all p_j where p_j is a companion of p_i, i.e., $p_i, p_j \in I(tr_r), \exists r$ and $p_j \notin I(tr_l), l \neq r$, then $C \leftarrow C - \{p_j\}$.
End For;

3) Repeat step 2 until C = T.
4) Complement the obtained MF of the fuzzy propositions present in set T.
5) End.

C. Trace of the Algorithm

1) On transformation of the primal net (see Fig. 3), we obtain the dual net (see Fig. 4). We complement n_i at each place p_i and complement n_i' at the axioms of the dual net. Initialize $C = \{p_6, p_7, p_5\}$, and $T = \{p_1, p_2\}$.

2) a) Let $p_i = p_6 \in I(tr_2)$. All input places (p_6 and p_7) of tr_2 possess MFs. Therefore, we evaluate

$$\mathrm{FTT}_{2,3} = t(\overline{n_6'}, \overline{n_7'})^T \circ R_{2,3}$$
$$\text{where } R_{2,3} = (\overline{t(\overline{n_6}, \overline{n_7})})^T \, \Phi \, \overline{n_3}$$
$$\mathrm{FTT}_{2,4} = t(\overline{n_6'}, \overline{n_7'})^T \circ R_{2,4}$$
$$\text{where } R_{2,4} = (\overline{t(\overline{n_6}, \overline{n_7})})^T \, \Phi \, \overline{n_4}$$

b) $p_3 \in O(tr_2)$ only; so, $\overline{n_3} = \mathrm{FTT}_{2,3}$;
$p_4 \in O(tr_2) \cap O(tr_3)$;
Therefore, $\overline{n_4} = \mathrm{FTT}_{2,4} \vee \mathrm{FTT}_{3,4}$. Since $\mathrm{FTT}_{3,4}$ is not known, the computation of $\overline{n_4}$ is pending.
Since computation of $\overline{n_3}$ is over, $C = C \cup \{p_3\}$.

c) $p_i = p_6 \in I(tr_2)$ only, and $\mathrm{FTT}_{2,3}$ and $\mathrm{FTT}_{2,4}$ where $p_3, p_4 \in O(tr_2)$ are known; therefore, $C \leftarrow C - \{p_6\} = \{p_7, p_5, p_3\}$. Similarly as p_7 is a companion of p_6, i.e., $p_6, p_7 \in I(tr_2)$ only, that is, $p_6, p_7 \notin I(tr_j), j \neq 2$; then, $C \leftarrow C - \{p_7\} = \{p_3, p_5\}$.

3) $C = \{p_3, p_5\} \neq \{p_1, p_2\}$. Therefore, repeat from step 2.

2a) $p_i = p_5 \in I(tr_3)$; since all input places of tr_3 possess MFs,

$$\mathrm{FTT}_{3,4} = \overline{n_5'} \circ R_{3,4}$$
$$\text{where } R_{3,4} = (\overline{n_5})^T \, \Phi \, \overline{n_4}.$$

b) Since $p_4 \in O(tr_2) \cap O(tr_3)$,
$\therefore\ \overline{n_4} = \mathrm{FTT}_{2,4} \vee \mathrm{FTT}_{3,4}$.
$C = C \cup \{p_4\} = \{p_3, p_4, p_5\}$.

c) $p_i = p_5 \in I(tr_3)$ only and $\mathrm{FTT}_{3,4}$ is known, where $p_4 \in O(tr_3)$,
$\therefore\ C \leftarrow C - \{p_5\} = \{p_3, p_4\}$.

3. $C = \{p_3, p_4\} \neq T = \{p_1, p_2\}$. Therefore, repeat from step 2.

2a) $p_i = p_3 \in I(tr_1)$. Since all input places (p_3, p_4) of tr_1 possess MFs,
we evaluate

$$\mathrm{FTT}_{1,1} = t(\overline{n_3'}, \overline{n_4'})^T \circ R_{1,1}$$
$$\text{where } R_{1,1} = (\overline{t(\overline{n_3}, \overline{n_4})})^T \, \Phi \, \overline{n_1}$$
$$\mathrm{FTT}_{1,2} = t(\overline{n_3'}, \overline{n_4'})^T \circ R_{1,2}$$
$$\text{where } R_{1,2} = (\overline{t(\overline{n_3}, \overline{n_4})})^T \, \Phi \, \overline{n_2}.$$

b) $p_1 \in O(tr_1)$ only, so

$$\overline{n_1} = \mathrm{FTT}_{1,1};\ p_2 \in O(tr_1)$$

only, so $\overline{n_2} = \mathrm{FTT}_{1,2}$;
since *computation* of $\overline{n_1}, \overline{n_2}$ is over,

$$C = C \cup \{p_1, p_2\} = \{p_{3,}, p_4, p_1, p_2\}.$$

c) $p_i = p_3 \in I(tr_1)$ only, and computation of $\mathrm{FTT}_{1,1}$ and $\mathrm{FTT}_{1,2}$ is over where $p_1 \in O(tr_1)$ and $p_2 \in O(tr_1)$, $\therefore C \leftarrow C - \{p_3\} = \{p_4, p_1, p_2\}$.
As p_4 is a companion of p_3 and $p_3, p_4 \in I(tr_1)$ only, $\therefore C \leftarrow C - \{p_4\} = \{p_1, p_2\} = T$. Therefore, we move to step 4.

4) Complement the obtained MFs of the fuzzy propositions in set T to get n'_1 and n'_2.
5) End.

D. Time Complexity

Let us consider n chained primary rules, each with m fuzzy propositions in the antecedent. The dual Petri net constructed from the primal net thus would have n rules each with m fuzzy propositions in the consequent. Since there are n rules, we should have n transitions in both the primal and the dual net. Thus, we have $(n \times m)$ number of implication relational matrices, and therefore, $(m \times n)$ number of steps of abduction to infer the MF [54]. The overall time complexity of the algorithm thus is $O(mn)$, presuming a uniform cost for composition operation and t-norm computation for the antecedent.

V. Conclusion

The paper examines a new approach to automatic extraction of MF in fuzzy abduction by two-step extension of propositional contraposition property. The first extension lies in the generalization of propositional contraposition property by considering multiple propositions in antecedent and consequent of a rule, while the second extension adds the notion of fuzziness to the contraposition property. The paper offers two fundamental advantages over the existing literature on abduction. The existing GMT-based formulation of fuzzy abduction provides a single relation using t-norms on the MFs of the fuzzy propositions present in the antecedent of a rule, and thus, the MF of individual fuzzy propositions of the antecedent cannot be separately distinguished/obtained. This has been overcome in this paper with the help of fuzzy extension of the contraposition property. Alternative formulation of fuzzy abduction (designed with a view to providing a better solution) requires evaluation of inverse implication relations. The extensive computations required in evaluating fuzzy inverse relations [11] (with respect to max-min composition operator) to obtain abductive inferences can be avoided by the proposed approach.

Special emphasis is given here to retrieval of MFs due to abduction. Conditions for abductive retrieval for simple rules with single fuzzy proposition in both antecedent and consequent and complex rules with multiple propositions in the consequent have been derived. Theorem 7 provides an interesting observation, which holds for any implication function that supports the fuzzy contraposition property. It states that for a given fuzzy rule "*if x is A, then y is B*," if B'_i is the observed MF and A'_i is the inferred MF, then if $B'_1 \geq B'_2$, we obtain $A'_1 \geq A'_2$.

Finally, this paper demonstrates the scope of abduction through multiple chained rules with the help of an additional structure similar to that of a Petri net. The algorithm used for abduction first transforms a primal network of chained rules to its dual form and then performs reasoning using classical generalized modus ponens. The resulting MFs of the inferred fuzzy propositions are complemented to get back the desired results. A complexity analysis reveals that for n rules each with m fuzzy propositions, we need to evaluate $(m \times n)$ number of relational matrices. Consequently, time complexity of the abductive reasoning algorithm is $O(mn)$, presuming a uniform cost for composition operation and t-norm computation for the antecedent.

Appendix

Theorem A1: Relational matrices R and R_i for $i = 1$ to n constructed by the Lukasiewicz implication function satisfy

$$R = \overset{m}{\underset{i=1}{S}} R_i(X_1, X_2, \ldots, X_n; Y_i).$$

Proof: With standard definitions of $\mu_{A_i}(x_i)$ and $\mu_{B_j}(y_j)$, we construct R by Lukasiewicz implication and thus obtain

$$\begin{aligned}
&= R(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m)\\
&= \mathrm{Min}\Big[1, \Big(1 - \overset{n}{\underset{i=1}{t}} \mu_{A_i}(x_i) + \overset{m}{\underset{j=1}{s}} \mu_{B_j}(y_j)\Big]\\
&= \overset{m}{\underset{j=1}{s}} \Big[\mathrm{Min}\Big\{1, \Big(1 - \overset{m}{\underset{i=1}{t}} \mu_{A_i}(x_i) + \mu_{B_j}(y_j)\Big)\Big\}\Big]\\
&= \overset{m}{\underset{j=1}{s}} R_i(x_1, x_2, \ldots, x_n; y_j).
\end{aligned}$$

Since the last result holds $\forall x_i \in X_i, i = 1$ to n, we obtain

$$R = \overset{m}{\underset{i=1}{S}} R_i(X_1, X_2, \ldots, X_n; Y_i).$$

□

Theorem A2: For $x_i \in X_i\ \forall i, y_1 \in Y_1, y_2 \in Y_2$ and a given $\overset{n}{\underset{\forall i}{T}}(A'_i)$

$$\overset{n}{\underset{\forall i}{T}}(A'_i) o (R_1 S R_2) = \overset{n}{\underset{\forall i}{T}}(A'_i) o R_1 S \overset{n}{\underset{\forall i}{T}}(A'_i) o R_2$$

holds when R_1 and R_2 are constructed by the Lukasiewicz implication function.

Proof: By the Lukasiewicz implication function, we have

$$\begin{aligned}
&R_1(x_1, x_2, \ldots, x_n; y_1)\\
&= \mathrm{Min}[1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) ... t \mu_{A_n}(x_n)) + \mu_{B_1}(y_1)]\\
&R_2(x_1, x_2, \ldots, x_n; y_2)\\
&= \mathrm{Min}[1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n)) + \mu_{B_2}(y_2)]\\
&\text{and } R_1(x_1, x_2, \ldots, x_n; y_1) s R_2(x_1, x_2, \ldots, x_n; y_2)\\
&= \mathrm{Min}[1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n))\\
&\quad + (\mu_{B_1}(y_1) s (\mu_{B_2}(y_2))].
\end{aligned}$$

Now, let $\overset{n}{\underset{i=1}{T}}(A'_i) = [a'_1 a'_2 \ldots]$.

Therefore

$$\begin{aligned}
&\overset{n}{\underset{\forall i}{T}}(A'_i) o (R_1 S R_2)\\
&= [a'_1 a'_2 \ldots] \circ [\mathrm{Min}\{1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots\\
&\quad \cdots t \mu_{A_n}(x_n)) + (\mu_{B_1}(y_1) s (\mu_{B_2}(y_2)\}], \quad \text{for } x_i \in X_i, \forall i\\
&\qquad \forall y_1 \in Y_1, y_2 \in Y_2\\
&= \underset{\forall i}{\vee} [a'_i \wedge \mathrm{Min}\{1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n))\\
&\quad + (\mu_{B_1}(y_1) s (\mu_{B_2}(y_2))\}], \quad \text{for } x_i \in X_i, \forall i\\
&\qquad \forall y_1 \in Y_1, y_2 \in Y_2
\end{aligned}$$

$$= \underset{\forall i}{\vee}[a_i' \wedge [\mathrm{Min}\{1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t\mu_{A_n}(x_n))$$
$$+ (\mu_{B_1}(y_1)\}]]s$$
$$\underset{\forall i}{\vee}[a_i' \wedge [\mathrm{Min}\{1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t\mu_{A_n}(x_n))$$
$$+ (\mu_{B_2}(y_2)\}]], \quad \text{for } x_i \in X_i \ \forall i$$
$$\forall\, y_1 \in Y_1 \text{ and } y_2 \in Y_2$$
$$= [a_1' a_2' \ldots] \circ R_1 S [a_1' a_2' \ldots] \circ R_2$$
$$= \overset{n}{\underset{i=1}{T}} (A_i') o R_1 S \overset{n}{\underset{i=1}{T}} (A_i') o R_2.$$

Theorem A3: Rule 1 is equivalent to Rule 3 when the implication relations for these rules are constructed by the Lukasiewicz implication function.

Proof: Let $R_1(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m)$ and $R_3(y_1, y_2, \ldots, y_m; x_1, x_2, \ldots, x_n)$ be the elements of the implication relations for Rules 1 and 3, respectively, constructed by the Lukasiewicz implication function. Therefore

$$R_1(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m)$$
$$= \mathrm{Min}\Big[1, \Big\{1 - \overset{n}{\underset{i=1}{t}} \mu_{A_i}(x_i) + \overset{m}{\underset{j=1}{s}} \mu_{B_j}(y_j)\Big\}\Big] \tag{A1}$$

$$\mathrm{R}_3(y_1, y_2, \ldots, y_m; x_1, x_2, \ldots, x_n)$$
$$= \mathrm{Min}\Big[1, \Big\{1 - \overset{m}{\underset{j=1}{t}} \mu_{\overline{B_j}}(y_j) + \overset{n}{\underset{i=1}{s}} \mu_{\overline{A_i}}(x_i)\Big\}\Big]$$
$$= \mathrm{Min}\Big[1, \Big\{1 - (1 - \overset{m}{\underset{j=1}{s}} \mu_{B_j}(y_j)) + (1 - \overset{n}{\underset{i=1}{t}} \mu_{A_i}(x_i))\Big\}\Big]$$

(by De Morgan's Law)

$$= \mathrm{Min}\Big[1, \Big\{1 - \overset{n}{\underset{i=1}{t}} \mu_{A_i}(x_i) + \overset{m}{\underset{j=1}{s}} \mu_{B_j}(y_j)\Big\}\Big]$$
$$= R_1(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) \text{ (by A1)}.$$

□

Proof of Theorem 11

Consider the contraposition rule "if y is $\overline{B}$ and z is $\overline{C}$, then x is $\overline{A}$." We use the Lukasiewicz implication rule to construct the relation R. Let $\overline{A} = [\overline{a_i}]_{1\times n}$, $\overline{B} = [\overline{b_j}]_{1\times m}$, $\overline{C} = [\overline{c_k}]_{1\times p}$, and $R = [r_{jk,i}]$, where

$$r_{jk,i} = \mathrm{Min}[1, 1 - (\overline{\mathrm{b_j}} \wedge \overline{c_k}) + \overline{a_i}] \quad \forall j, k, i.$$

We infer $A' = [a_i']_{1\times n}$, where

$$\overline{a_i'} = \underset{\forall j,k}{\vee} (\overline{b_j'} \wedge \overline{c_k'}) \wedge [1 \wedge \{1 - (\overline{b_j} \wedge \overline{c_k}) + \overline{a_i}\}]$$
$$= \underset{\forall j,k}{\vee} (\overline{b_j} \wedge \overline{c_k}) \wedge [1 \wedge \{1 - (\overline{b_j} \wedge \overline{c_k}) + \overline{a_i}\}]$$

($\because$ $\overline{b_j'} = \overline{b_j}, \overline{c_k'} = \overline{c_k}$ is required for abductive retrieval)

$$= (\overline{b_u} \wedge \overline{c_v}) \wedge [1 \wedge \{1 - (\overline{b_u} \wedge \overline{c_v}) + \overline{a_i}\}] \vee$$
$$\left\{ \underset{\substack{\forall j,k \\ j\neq u \\ k\neq v}}{\vee} (\overline{b_j} \wedge \overline{c_k}) \wedge [1 \wedge \{1 - (\overline{b_j} \wedge \overline{c_k}) + \overline{a_i}\}] \right\}$$
$$= \overline{a_i}$$

if the following conditions jointly hold:

$$(\overline{b_u} \wedge \overline{c_v}) \wedge [1 \wedge \{1 - (\overline{b_u} \wedge \overline{c_v}) + \overline{a_i}\}]$$
$$\geq \underset{\substack{\forall j,k \\ j\neq u \\ k\neq v}}{\vee} (\overline{b_j} \wedge \overline{c_k}) \wedge [1 \wedge \{1 - (\overline{b_j} \wedge \overline{c_k}) + \overline{a_i}\}] \tag{A2}$$
$$b_u = 0 \text{ and } c_v = 0. \tag{A3}$$

Substituting (A3) into (A2), we have

$$\overline{a_i} \geq \underset{\substack{\forall j,k \\ j\neq u \\ k\neq v}}{\vee} [(\overline{b_j} \wedge \overline{c_k}) \wedge [1 \wedge \{1 - (\overline{b_j} \wedge \overline{c_k}) + \overline{a_i}\}] \tag{A4}$$

which holds if

$$(\overline{b_j} \wedge \overline{c_k}) \geq [1 \wedge \{1 - (\overline{b_j} \wedge \overline{c_k}) + \overline{a_i}\}]$$
$$\forall j, k, j \neq u, k \neq v. \tag{A5}$$

Substituting (A5) into (A4), we have

$$\overline{a_i} \geq \underset{\substack{\forall j,k \\ j\neq u \\ k\neq v}}{\vee} [1 \wedge \{1 - (\overline{b_j} \wedge \overline{c_k}) + \overline{a_i}\}]. \tag{A6}$$

Inequality (A6) holdsif either $1 - (\overline{b_j} \wedge \overline{c_k}) \leq 0$ which yields $b_j = c_k = 0, \forall j, k, j \neq u, k \neq v$ and thus is restrictive, or

$$1 - (\overline{b_j} \wedge \overline{c_k}) + \overline{a_i} \geq 1$$
$$\Rightarrow \overline{a_i} \geq (\overline{b_j} \wedge \overline{c_k}) \quad \forall j, k, j \neq u, k \neq v. \tag{A7}$$

From (A3) and (A7), we thus have the condition of abductive retrieval, rewritten as

1) $b_u = c_v = 0$.
2) $\overline{a_i} \geq \overline{b_j} \wedge \overline{c_k} \forall j, k, j \neq u, k \neq v$. □

Proof of Theorem 12

Consider the rule "if y is $\overline{B}$ and z is $\overline{C}$, then x is $\overline{A}$." Let $\overline{A} = [\overline{a_i}]_{1\times n}$, $\overline{B} = [\overline{b_j}]_{1\times m}$, and $\overline{C} = [\overline{c_k}]_{1\times p}$.

We construct the relational matrix $R = [r_{jk,i}]$ using Diens–Rescher implication and presume observed MFs $B' = B$ and $C' = C$ and thus obtain A' by extended contraposition property. Here

$$\overline{a_i'} = \underset{\forall j,k}{\vee} (\overline{b_j'} \wedge \overline{c_k'}) \wedge r_{jk,l}$$
$$= \underset{\forall j,k}{\vee} (\overline{b_j'} \wedge \overline{c_k'}) \wedge (\overline{(\overline{b_j} \wedge \overline{c_k})} \vee \overline{a_i})$$
$$= \underset{\forall j,k}{\vee} (\overline{b_j} \wedge \overline{c_k}) \wedge (\overline{(\overline{b_j} \wedge \overline{c_k})} \vee \overline{a_i})$$

(as for abductive retrieval, we set $\overline{b_j'} = \overline{b_j}$ and $\overline{c_k'} = \overline{c_k}$)

$$= \{(\overline{b_u} \wedge \overline{c_v}) \wedge (\overline{(\overline{b_u} \wedge \overline{c_v})} \vee \overline{a_i})\} \vee$$
$$\left\{ \underset{\substack{\forall j,k \\ j\neq u \\ k\neq v}}{\vee} (\overline{b_j} \wedge \overline{c_k}) \wedge (\overline{(\overline{b_j} \wedge \overline{c_k})} \vee \overline{a_i}) \right\}$$
$$= \overline{a_i}. \tag{A8}$$

If the following conditions hold:

$$(\overline{b_u} \wedge \overline{c_v}) \wedge ((\overline{\overline{b_u} \wedge \overline{c_v}}) \vee \overline{a_i})$$
$$\geq (\overline{b_j} \wedge \overline{c_k}) \wedge ((\overline{\overline{b_j} \wedge \overline{c_k}}) \vee \overline{a_i}), \forall j, k, j \neq u, k \neq v \quad \text{(A9)}$$
$$b_u = 0, \quad c_v = 0. \quad \text{(A10)}$$

Substituting (A10) into (A9), we have

$$\overline{a_i} \geq (\overline{b_j} \wedge \overline{c_k}) \wedge ((\overline{\overline{b_j} \wedge \overline{c_k}}) \vee \overline{a_i}) \quad \forall j, k, j \neq u, k \neq v \quad \text{(A11)}$$

which holds if

$$(\overline{b_j} \wedge \overline{c_k}) \leq \overline{a_i}. \quad \text{(A12)}$$

Therefore, from (A10) and (A12), we have the conditions for abductive retrieval, given by

1) $b_u = 0, c_v = 0$ for any *u*, *v*.
2) $(\overline{b_j} \wedge \overline{c_k}) \leq \overline{a_i}, \forall j, k$, except $j = u, k = v$. □

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