Extending the Contraposition Property of Propositional Logic for Fuzzy Abduction

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Abstract—Abduction deals with assumption-based reasoning to explain an observation. In the context of fuzzy reasoning, abduction attempts to determine the membership function of the fuzzy propositions present in the antecedent of a rule when the membership functions for the propositions in the consequent of the rule are given. Currently available models of fuzzy abduction are capable of inferring the membership function of the antecedent clause accurately when the antecedent includes single fuzzy proposition. However, when the antecedent clause of a rule contains multiple fuzzy propositions, these models fail to determine the independent membership function of the individual propositions present in the antecedent. This paper presents a new formulation to handle such situations. An algorithm for automated abduction using the extended contraposition property has been developed to demonstrate the principle of abduction with rules containing one or more fuzzy propositions in the antecedent/consequent. The time complexity of the proposed fuzzy abduction for a sequence of \( n \) chained rules, where each rule has \( n \) fuzzy propositions, is \( O(mn) \), considering a uniform cost for composition operation and \( t \)-norm computation of the antecedent.

Index Terms—Contraposition property, fuzzy abduction, fuzzy logic network (FLN), implication relations, propositional logic.

I. INTRODUCTION

A

B

DUCTION refers to an assumption-based reasoning strategy to explain an observation [3]. The principle of abduction aims to determine a subset from a given set of assumed premises, which, in conjunction with a given knowledge base, is capable of explaining the observation. For example, in a diagnostic problem, we need to identify a defective component from a list of possible defects \( d_1, d_2, \ldots, d_n \) to support a set of observations about malfunctioning of the system using a given set of diagnostic rules.

Several attempts have been made to handle the abduction problem [4], [6]–[12], [14]–[17], [21]–[23], [28], [29], [31], [32], [35], [37], [39]–[42]. Most of the techniques on abduction can be broadly classified into two major heads: causal [19] and implicational [1] reasoning. Although treated similarly in our regular use, causality and implication are different in the context of reasoning, as the former requires a temporal ordering between the antecedent and the consequent, while the latter does not require such ordering [45]. In a simplistic causal reasoning [46] problem, \( P \) is a cause of \( Q \), \( Q \) is observed, and abduction refers to inferring \( P \). Researchers employed probabilistic frameworks to handle the causal abduction. One of the well-known works in this regard is the belief revision paradigm in a Bayesian network. Pearl’s belief revision model [47] in a causal framework is possibly the first successful work in this regard. Several extensions to Pearl’s model have been undertaken, each with a new flavor [7], [8], [10].

Similarly, in implicational reasoning, abduction refers to extraction of \( P \) from the rule: if \( P \) then \( Q \), and the observation \( Q \). As the logic of propositions/predicates is incompetent to handle abduction [48], the early works on implicational abduction started with nonstandard logic. These logics fundamentally differ from their classical first-order (propositional/predicate logic) [49] counterpart by their inherent capability of nonmonotonic [50], [51] reasoning. It may be added here that an inferred statement in the context of nonmonotonic reasoning may sometimes contradict its predecessors/premises. Unless such contradiction is detected, we can infer \( P \) from the fact “\( \neg Q \)” and the given rule “if \( P \) then \( Q \)”.

In case a contradiction is found between the inferred \( P \) and the given/inferred set of statements, \( P \) is withdrawn. For example, given a default rule “if \( x \) is a bird, then \( x \) can fly” and an observation “\( x \) flies” makes someone guess that \( x \) is a bird. This assumption-based default abduction [30] is valid until it is discovered (by other defaults or through observations) that \( x \) is a bat (and not a bird). Several nonmonotonic and default logic models [29], [52], [53], [61] have been employed to handle abduction. Modal logic is also considered in the literature of abduction [21].

The logic of fuzzy sets [44] offers interesting solutions to the principles of abduction [2], [38]. In general, fuzzy abduction attempts to infer the membership function (MF) [62] of the antecedent, when the MFs of the consequent and the implication relation between the antecedent and the consequent of a rule are known. One primitive approach to construct a solution to this problem is to design relational equations involving known MF of the consequent and unknown antecedent and then to provide a relational algebraic approach to solve the problem. The abduction problem thus is translated into fuzzy relational equations. The early works on fuzzy abduction employed transpose the
implicational relation to derive fuzzy premises of a given rule [18], [19]. The transpose-based solution is advantageous for its simplicity in approach and insignificant computational complexity. However, the quality of transpose-based solution is not always acceptable [36]. Recently, researchers, instead of directly solving the fuzzy relational equation [24]–[27], employ a heuristic algorithm to compute max-min compositional inverse fuzzy relations [36] and offer interesting solutions to fuzzy abduction using the inverse relation. More recently, an alternative approach to computing the optimal inverse (fuzzy) implicational relation is proposed in [11] in connection with the abduction problem. This approach [11] seems to have less complexity than the approach in [36] and thus can effectively be used for abduction.

Here, we propose an alternative approach to handle the fuzzy abduction problem using an extension of the well-known contraposition property of propositional logic. For the sake of convenience, it may be added here that transforming a given rule “if \( p \) then \( q \)” to its equivalent form “if \( \neg q \) then \( \neg p \)” is the basis of propositional contraposition. When the antecedent is a conjunction of atomic propositions \( p_1, p_2, \ldots, p_n \) and the consequent is a disjunction of atomic propositions \( q_1, q_2, \ldots, q_m \), we transform a given primal rule to its equivalent dual form with modified antecedent as the conjunction of \( \neg q_j \)'s and modified consequent as the disjunction of \( \neg p_i \)'s [64]. This transformation has been referred to as the extended contraposition property. The transformed rule asserts falsehood of at least one \( p_i \), given all the \( q_j \)'s are false. It may be noted that the transformed rule is an explicit representation of the propositional backward chaining principle [55] applied on the original rule.

In this paper, we study the scope of fuzzy abduction by extending the contraposition property further in the context of fuzzy logic. We have shown that the extended contraposition property holds good in fuzzy logic, if the implication used in the fuzzy production rules is realized by Diens–Rescher- or Lukasiewicz-type implication functions. To illustrate the fuzzy extension of the contraposition property, consider \( x \) and \( y \) as two linguistic variables with universes \( U \) and \( V \), respectively. Let \( A \) and \( B \) be fuzzy subsets of \( U \) and \( V \), respectively; in other words, \( A \) and \( B \) are linguistic values of \( x \) and \( y \), respectively. The contraposition property in the context of fuzzy logic transforms a primal rule “if \( x \) is \( A \) then \( y \) is \( B \)” into its equivalent dual form “if \( y \) is \( \neg B \), then \( x \) is \( \neg A \),” where the bar above a fuzzy set denotes its complement. Therefore, if we know the MF of \( y \) is \( B' \) (\( B' \approx B \)), we can derive the MF of \( x \) is \( A' \) (\( A' \approx A \)) using the dual rule.

It may be mentioned here that generalized modus tollens (GMT)-based fuzzy abduction with a primal rule “if \( x_1 \) is \( A_1 \) and \( x_2 \) is \( A_2 \), then \( y_1 \) is \( B_1 \) or \( y_2 \) is \( B_2 \),” where the symbols have standard meaning, returns a single relation involving \( t \)-norm of the MFs of the premises \( x_i \) is \( A_i' \) for all \( i \) together, when the MFs of \( y_j \) is \( B_j' \) for all \( j \) are given. Therefore, computation of the individual MF of \( x_i \) is \( A_i' \) for all \( i \) separately becomes an indeterminate task in GMT-based fuzzy abduction. The fuzzy extension of the propositional contraposition property overcomes this limitation, as the conjunction of the premises, \( x_i \) is \( A_i \), of the primal rule now appears as disjunction of their complements in the consequent part of the dual rule. Therefore, given the MF of \( y_j \) is \( B_j' \) for all \( j \), we can evaluate the MF for \( x_i \) is \( \neg A_i' \) for all \( i \) by a generalized (fuzzy) modus ponens.

Such abductive interpretation has applications in many engineering problems, such as diagnosis and historical time-series prediction [33], [34].

A question naturally arises: How good is contraposition-based fuzzy abduction. To study the qualitative performance of the proposed fuzzy abduction, we consider retrieval of MF as an important measure. Retrieval here means that the computed MF \( x_i \) is \( \hat{A}_i \) will be equal to the given MF \( x_i \) of the antecedent for all \( i \), when the observed MF \( y_j \) is \( \hat{B}_j \) and the given MF \( y_j \) is \( B_j \) of the consequent are equal for all \( j \). Retrieval, however, requires satisfaction of specific conditions involving the MFs of \( x_i \) is \( A_i \), \( y_j \) is \( B_j \), and \( y_j \) is \( B_j' \). The conditions for abductive retrieval for Diens–Rescher- and Lukasiewicz-based implication have been derived here.

Reasoning through chained sequence of rules is an important issue in fuzzy logic [43]. This has been undertaken in this paper first by representing the chained rules as a fuzzy inferential network and then by transforming the rules in the network from their primal to dual form using extended contraposition property. Given the observed MF of the terminal [54] fuzzy propositions in the primal network, the principle of abduction can be applied in a layerwise fashion in the dual network, starting computation in a layer with known MFs and continuing computation in a layerwise manner until the MFs of the desired propositions are obtained. An algorithm for automated abduction in a fuzzy logic network (FLN) is presented following the principle outlined above. Reasoning efficiency [54] of the proposed algorithm depends greatly on the number of chained rules and the number of antecedent clauses in the rules. For an \( n \)-chain ruled system with \( m \) fuzzy propositions per rule, the time complexity of the algorithm is \( O(mn) \).

The rest of this paper is organized into five sections. In Section II, we extend the contraposition property of propositional logic and construct a formulation of fuzzy abduction using the extension. In Section III, we derive the conditions for abductive retrieval for the Lukasiewicz- and Diens–Rescher-type implication functions. Here, we also derive the condition for abductive retrieval when the formulation is undertaken as GMT. Section IV provides an algorithm for automated abductive reasoning with multiple chained rules using the dual FLN. Conclusions are listed in Section V.

II. CONTRAPOSITION-BASED FUZZY ABDUCTION

In this section, we study two important fuzzy extensions of propositional properties: 1) decomposition and 2) contraposition.

A. Fuzzy Decomposition

Let \( p, q, r, \) and \( s \) be four atomic (binary valued) propositions. In propositional logic, the rule “\( p \land q \rightarrow r \lor s \)” can equivalently be expressed as two rules “\( p \land q \rightarrow r \)” and “\( p \land q \rightarrow s \),” where \( \land, \lor, \) and \( \rightarrow \) denote AND, OR, and implication operators, respectively. We would now like to examine whether the above property holds in the logic of fuzzy sets.

Let \( X_i \) and \( Y_j \) be universes of fuzzy linguistic variables \( x_i \) and \( y_j \), respectively, for \( i = 1 \) to \( n \) and \( j = 1 \) to \( m \). Let \( A_i \) and \( B_j \) be fuzzy subsets of \( X_i \) and \( Y_j \), respectively.
Let $B_1$ be fuzzy sets in $X_i$ and $Y_j$, respectively. Consider a fuzzy (primal) rule:

Rule 1: If $x_1$ is $A_1$ and $x_2$ is $A_2$ and ... and $x_n$ is $A_n$, then $y_1$ is $B_1$ or $y_2$ is $B_2$ or ... or $y_m$ is $B_m$.

We would now present one interesting decomposition property of the rule, by which the rule would be represented as $m$ subrules connected by OR operators, as indicated below.

Rule 2: (If $x_1$ is $A_1$ and $x_2$ is $A_2$ and ... and $x_n$ is $A_n$, then $y_1$ is $B_1$) or

(If $x_1$ is $A_1$ and $x_2$ is $A_2$ and ... and $x_n$ is $A_n$, then $y_2$ is $B_2$) or

... ... ... ...

... (If $x_1$ is $A_1$ and $x_2$ is $A_2$ and ... and $x_n$ is $A_n$, then $y_m$ is $B_m$).

Equivalence of any two fuzzy rules can be established by testing equality of their fuzzy implication relations [27]. We would now examine the equivalence of Rules 1 and 2 using Diens–Rescher and Lukasiewicz implication functions. Let $t$ and $s$ be scalar $t$- and $s$-norm operators [18]. The $t$-norm (here, min) and $s$-norm (here, max) between two membership values $\mu_{A_i}(x_i) = a_i$ and $\mu_{B_j}(y_j) = b_j$ are given by

\[
\mu_{A_i}(x_i) \land \mu_{B_j}(y_j) = \{a_i \land b_j\} \\
\mu_{A_i}(x_i) \lor \mu_{B_j}(y_j) = \{a_i \lor b_j\}.
\]

While computing the $t$- or $s$-norms of two MFs, a specific order of the indices $i$ and $j$ is to be considered. Here, for each integer $i$, we consider all $j$ in ascending order and then consider the next integer $i$ in ascending order until $i = n$ and $j = m$. The cumulative $t$-norm of three or more MFs $\mu_{A_1}, \mu_{A_2}, \mu_{A_3}, \ldots, \mu_{A_n}$, where $\mu_{A_i}$ is defined on $X_i$, yields an $n$-dimensional relation, say, $R$, given by $R = \sum_{i=1}^{n} (A_i)$, where $A_i$ is a vector of membership values representing a fuzzy set defined on $X_i$ and an element of $R$ is defined as

\[
R(x_1, x_2, \ldots, x_n | x_i \in X_i; i = 1, \ldots, n) = (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n)) \\
= (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n)) \ldots \\
= \sum_{i=1}^{n} \mu_{A_i}(x_i).
\]

Similarly, cumulative $s$-norm of $\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)$ is denoted by $\sum_{i=1}^{n} (A_i)$. On the other hand, the cumulative $s$-norm among $n$ MFs $A_i$, denoted by $\sum_{i=1}^{n} (A_i)$, is an $n$-dimensional relation which can be evaluated by replacing $t$ by $s$ in (1).

We now define an $S$-norm operator between two relational matrices $R_1 (X_1, X_2, \ldots, X_n; Y_1)$ and $R_2 (X_1, X_2, \ldots, X_n; Y_2)$, which yields a new relational matrix $R_{12} (X_1, X_2, \ldots, X_n; Y_1, Y_2)$; the row indices of the resulting matrix is the joint occurrence of the valuation space of $x_1, x_2, \ldots, x_n$, where the column indices represent the joint occurrences of $y_1, y_2$. Let $R_{12} (x_1, x_2, \ldots, x_n; y_1, y_2)$, $y_1, y_2 \in (Y_1 \times Y_2)$ be an element of $R_{12}$, which is obtained by applying an $s$-norm on $R_1 (x_1, x_2, \ldots, x_n; y_1)$ and $R_2 (x_1, x_2, \ldots, x_n; y_2)$, which are elements of $R_1$ and $R_2$, respectively. Thus

\[
R_{12} (x_1, x_2, \ldots, x_n; y_1, y_2) = R_1 (x_1, x_2, \ldots, x_n; y_1) s R_2 (x_1, x_2, \ldots, x_n; y_2)
\]

for $y_1 \in Y_1$ and $y_2 \in Y_2$; $s$ is a scalar $s$-norm [18].

If the cardinalities of $Y_1$ and $Y_2$ are $m_1$ and $m_2$, respectively, then to define the entire relation $R_{12}$, for each possible $m_1 m_2$ pairs, the above $s$-norm operation is to be computed. For notational simplicity, we denote

\[
R_{12} (X_1, X_2, \ldots, X_n; Y_1, Y_2) = R_{12},
\]

say

\[
S \left( \begin{array}{c} R_1 (x_1, x_2, \ldots, x_n; y_1), (R_2 (x_1, x_2, \ldots, x_n; y_2) \end{array} \right).
\]

This notation will help us in proving some results later. We call $S$ an $S$-norm over two relations, which is realized by applying the $\lor$ operator on the columns of $R_1$ and $R_2$ for $y_1, y_2 \in Y_1 \times Y_2$ in an ordered manner as defined above. Here, for a given $y_1$, the above $S$-norm operation is performed for all $y_2$ in a fixed order, considering a fixed instantiation of $x_1, x_2, \ldots, x_n$ in $X_1 \times X_2 \times \cdots \times X_n$ for both $R_1$ and $R_2$. Note that we can repeatedly apply the above $S$-norm to define $R_{123}$ from $R_{12}$ and $R_3$, and so on. We now provide an example to illustrate the computation of $R_{12}$.

Example 1: Given $X_1 = \{1, 2\}, X_2 = \{3, 4\}, Y_1 = \{5, 6\}$, and $Y_2 = \{7, 8\}$. Let $x_1 \in X_1$ and $y_1 \in Y_1$ for $i = 1$. Let $\mu_{A_i}(X_1) = \{0.1/0.7\}$, $\mu_{A_2}(X_2) = \{0.3/0.9/4\}$, $\mu_{B_i}(Y_1) = \{0.2/5/0.8/6\}$, and $\mu_{B_i}(Y_2) = \{0.6/7/0.9/8\}$.

We now construct relations $R_1$ and $R_2$ for the Rules 1 and 2, respectively, using the Lukasiewicz implication function, where Rule 1 is “if $x_1$ is $A_1$ and $x_2$ is $A_2$, then $y_1$ is $B_1$” and Rule 2 is “if $x_1$ is $A_1$ and $x_2$ is $A_2$, then $y_2$ is $B_2$.

$$
\begin{array}{c}
x_1, x_2 \ y_1 \ 5 \ 6 \\
1, 3 \ [1.0/1.0] \\
1, 4 \ [1.0/1.0] \\
2.3 \ [0.9/1.0] \\
2.4 \ [0.5/1.0]
\end{array}
\begin{array}{c}
x_1, x_2 \ y_2 \ 7 \ 8 \\
1, 3 \ [1.0/1.0] \\
1, 4 \ [1.0/1.0] \\
2.3 \ [1.0/1.0] \\
2.4 \ [0.9/1.0]
\end{array}
\]

Now, we construct $R_{12} = R_1 S R_2$.

$$
\begin{array}{c}
x_1, x_2 \ y_1 \ y_2 \ 5, 7 \ 5, 8 \\
1, 3 \ [1.0/1.0] \\
1, 4 \ [1.0/1.0] \\
2.3 \ [0.9/1.0] \\
2.4 \ [0.5/1.0]
\end{array}
\begin{array}{c}
x_1, x_2 \ y_1 \ y_2 \ 5, 7 \ 5, 8 \\
1, 3 \ [1.0/1.0] \\
1, 4 \ [1.0/1.0] \\
2.3 \ [0.9/1.0] \\
2.4 \ [0.5/1.0]
\end{array}
\]

Note that $R_{12}$, in general, should be a 4-D relation of size $2 \times 2 \times 2 \times 2$. However, we represented it as a 2-D relation of size $4 (= 2 \times 2) \times 4 (= 2 \times 2)$ for the sake of convenience in reasoning using max-min composition operation [19].

The $S$-norm of $R_1 (X_1, X_1, \ldots; Y_1)$ and $R_2 (X_1, X_1, \ldots; Y_2), \ldots, R_m (X_1, X_1, \ldots; Y_m)$ is obtained in $(m - 1)$ steps
with intermediate relation \( R_{12}, R_{123}, \ldots, R_{123 \ldots (m-1)}, \) where
\[
\begin{align*}
R_{12} &= S \{ (R_1(x_1, x_2, \ldots, x_n; y_1) \} \\
R_{123} &= S \{ R_{12}, R_3 \} \\
R_{123m} &= S \{ R_{123}, R_4 \}
\end{align*}
\]
and so on, and finally
\[
R_{123 \ldots m} = \bigcup_{y_1, y_2, \ldots, y_{m-1} \in Y_1 \times Y_2 \times \cdots \times Y_{m-1}} \{ R_{123 \ldots (m-1)}, R_m \}.
\]

When \( R_1, R_2, \ldots, R_m \) are matrix relations, we write \( R_{123 \ldots m} = R_1SR_2 \ldots SR_m = S_{i=1}^m R_i \), say, where the computation of the successive \( S \)-norms is performed in order of their occurrence. Let \( R(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) \) and \( R_i(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) \) be the implication relations for Rule 1 and subrule \( i \) under Rule 2, respectively. Theorem 1 indicates that \( R = S_{i=1}^m R_i \) holds well by the Diens–Rescher implication function.

**Theorem 1:** Let \( R \) and \( R_i \), respectively, be the relations for Rule 1 and subrule of Rule 2 for \( i = 1 \) to \( n \) constructed by the Diens–Rescher implication function. Then, \( R = S_{i=1}^m R_i(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) \).

**Proof:** Let \( \mu_{A_i}(x_i) \) be the MF for \( x_i \) is \( A_i \), for \( i = 1 \) to \( n \). Then, by Diens–Rescher implication, we have
\[
\begin{align*}
R(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) &= 1 - (\mu_{A_1}(x_1) \mu_{A_2}(x_2) \cdots \mu_{A_n}(x_n)) \\
&= s(\mu_{B_1}(y_1) \mu_{B_2}(y_2) \cdots \mu_{B_m}(y_m)) \\
&= \left\{ \left( 1 - \frac{1}{t} \mu_{A_1}(x_1) \right) \mu_{B_1}(y_1) \right\} s \left\{ \left( 1 - \frac{1}{t} \mu_{A_2}(x_2) \right) \mu_{B_2}(y_2) \right\} s \cdots \left\{ \left( 1 - \frac{1}{t} \mu_{A_n}(x_n) \right) \mu_{B_n}(y_m) \right\}.
\end{align*}
\]
Therefore, \( R(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) \)
\[
\begin{align*}
R_1(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) &= R_1(x_1, x_2, \ldots, x_n; y_1) s \\
R_2(x_1, x_2, \ldots, x_n; y_2) s \cdots s R_m(x_1, x_2, \ldots, x_n; y_m) \\
&= \frac{m}{s} R_j(x_1, x_2, \ldots, x_n; y_j).
\end{align*}
\]
Since (2) holds \( \forall x_i \in X_i, i = 1 \) to \( n \), we obtain
\[
R = \bigcup_{i=1}^m R_i(x_1, x_2, \ldots, x_n; y_i)
\]
thereby proving the statement of Theorem 1.

Now, the inference obtained by Rule 1 is given by
\[
T(T(A'_1, A'_2), \ldots, A'_n) \circ R = \bigcup_{i=1}^n T(A'_i) \circ R,
\]
where \( T(A'_1, A'_2) \) is the \( t \)-norm between two vectors \( A'_1 \) and \( A'_2 \), resulting in a matrix whose \((k \times l)\)th component is the \( t \)-norm of \((\mu_{A'_1}(x_i^{(k)}), \mu_{A'_2}(x_j^{(l)}))\), and \( \circ \) denotes the max–min composition operator, which acts like a typical matrix multiplication operator with the sum and the product being replaced by max and min operations, respectively. Here, \( \mu_{A'_1}(x_i^{(k)}) \) and \( \mu_{A'_2}(x_j^{(l)}) \) denote the \( k \)th and \( l \)th component of \( A'_1 \) and \( A'_2 \), respectively. Theorem A1 in the Appendix shows that the result in Theorem 1 is also true for Lukasiewicz implication function.

Theorem 2 provides an interesting observation that decom- position of Rule 1 into the subrules of Rule 2 does not lose any information of Rule 1, as we can reconstruct the inference generated by Rule 1 by taking \( S \)-norm of the inferences obtained from Rule 2. Given \( T(A'_i) \), and suppose that \( R_1(X_1, X_2; Y_1) \) and \( R_2(X_1, X_2; Y_2) \) are constructed with Lukasiewicz implication function, we by Theorem A2 given in the Appendix can prove the following statement:
\[
\bigcup_{i=1}^n (A'_i) \circ R(T_{SR_1}T(A'_i) \circ R_2) = \bigcup_{i=1}^n (A'_1) \circ R_1 \bigcup_{i=1}^n (A'_2) \circ R_2 = S_{i=1}^m \{ T(A'_i) \circ R_1, T(A'_i) \circ R_2 \}.
\]

The above result is used to prove Theorem 2. It is important to note that the above expression also holds, when \( R_1 \) and \( R_2 \) are constructed with Diens–Rescher implication.

**Theorem 2:** For Diens–Rescher- or Lukasiewicz-type implication, the inference obtained by Rule 1 is the OR of the inference obtained by the \( j \)th subrule of Rule 2 for \( j = 1 \) to \( m \).

**Proof:** For Diens–Rescher- and Lukasiewicz-type implications, by (2), we have
\[
R = \bigcup_{y_1, y_2, \ldots, y_m \in Y_m} \bigcup_{x_1, x_2, \ldots, x_n \in X_m} \{ (R_1(x_1, x_2, \ldots, x_n; y_1), \ldots, (R_m(x_1, x_2, \ldots, x_n; y_m)) \}.
\]
Now, the inference obtained by Rule 1 is given by
\[
T_i R \bigcup_{y_1, y_2, \ldots, y_m \in Y_m} \bigcup_{x_1, x_2, \ldots, x_n \in X_m} \{ (R_1(x_1, x_2, \ldots, x_n; y_1), \ldots, (R_m(x_1, x_2, \ldots, x_n; y_m)) \}.
\]
Theorem 3: Rule 1 is equivalent to Rule 3, when the implication relations for these rules are constructed by the Diens–Rescher implication function.

Proof: Let \( R_1(x_1, X_1, x_2, \ldots, x_n, Y_1, Y_2, \ldots, Y_m) \) and \( R_3(y_1, Y_1, x_2, \ldots, x_n; X_1, X_2, \ldots, X_n) \) be the implication relations for Rules 1 and 3, respectively, constructed by the Diens–Rescher implication function. Then, with the defined \( t \)- and \( s \)-norm operators, we obtain the \((x_1, \ldots, x_n, y_1, \ldots, y_m) \)-th positional element of \( R_1 \) and \((y_1, \ldots, y_m, x_1, \ldots, x_n) \)-th positional element of \( R_3 \) and test their equality for all possible choices of \((x_1, \ldots, x_n, y_1, \ldots, y_m) \). Here

\[
R_1(x_1, \ldots, x_n, y_1, \ldots, y_m) = 1 - \min \left\{ \mu_{A_j}(x_j) \right\} \left( \max \left\{ \mu_{B_j}(y_j) \right\} \right) \quad (5)
\]

and

\[
R_3(y_1, \ldots, y_m, x_1, \ldots, x_n) = 1 - \max \left\{ \mu_{A_j}(x_j) \right\} \left( \min \left\{ \mu_{B_j}(y_j) \right\} \right)
\]

(by De Morgan’s theorem)

\[
R_1(x_1, \ldots, x_n, y_1, \ldots, y_m) \quad [\text{by (5)}]
\]

The equality of \( R_1(x_1, \ldots, x_n, y_1, \ldots, y_m) \) and \( R_3(y_1, \ldots, y_m, x_1, \ldots, x_n) \) for all choices of \( x_1, \ldots, x_n, y_1, \ldots, y_m \) ensures the equivalence of the two rules, thereby proving the theorem. \(\square\)

Theorem A3 in the Appendix shows that the property of Theorem 3 also holds for the Łukasiewicz implication function. It can be easily proved using Theorems 2 and 3 that given Rule 1, we can get its dual form decomposing it into \( n \) subrules of the form “if \( y_1 = B_1 \) and \( y_2 = B_2 \) and \( \ldots \) \( y_m = B_m \), then \( x_i = A_i' \) for \( i = 1 \) to \( n \) and derive the inference for \( x_i = A_i' \forall i \). This is the basis of contraposition-based fuzzy abduction.

In the subsequent sections, for the sake of simplicity, we would use \( \land \) and \( \lor \) instead of \( t \)- and \( s \)-norms, unless we essentially require them.

III. PROPERTIES OF ABDUCTIVE RETRIEVAL

In this section, we study a few interesting properties and conditions for “abductive retrieval” for two cases: first considering rules with single fuzzy proposition in antecedent and consequent, and later rules with multiple propositions in the consequent. While the former rules are considered for simplicity in analysis, the latter rules are taken up for the sake of completeness of the study. In the latter case, primal rules like Rule 1 with \( m \) fuzzy propositions in the consequent and \( n \) propositions in the antecedent are decomposed into \( n \) subrules each with one fuzzy proposition in consequent and \( m \) fuzzy propositions in the antecedent. This justifies the significance of primal rules with multiple propositions in the consequent.

A. Rules With Single Fuzzy Proposition in Antecedent and Consequent

Consider a fuzzy production rule “if \( x \) is \( A \), then \( y \) is \( B \)” where \( x \) and \( y \) are two linguistic variables, and \( A \) and \( B \) are
fuzzy sets are defined on their respective universes $U$ and $V$. By applying a fuzzy contraposition property, which is introduced in the previous section, we transform this primal rule into its equivalent dual form: if $y$ is $B$, then $x$ is $\overline{A}$. Now, given the MF for $y$ is $BF$, we want to derive the MF of $x$ is $A'$. Assuming that the MFs are available in vector form, i.e., $\overline{A} = [\overline{a}_i]_{1 \times n}$, $B = [b_j]_{1 \times m}$, and $BF = [BF_{ij}]_{1 \times m}$, by fuzzy modus ponens, we obtain

$$\overline{A} = BF \circ R$$  \hspace{1cm} (6)

where $R$ is the implication relation for the rule: if $y$ is $B$, then $x$ is $\overline{A}$. Retrieval here means getting back $A' = A$, when $B' = B$ is supplied. Substituting $B' = B$ into (6), we obtain

$$\overline{A} = BF \circ R.$$  \hspace{1cm} (7)

A desirable property would be to obtain $A' = A$, but it is not possible for every implication function. Now, setting different implication functions for $R$, we determine the condition for $\overline{A} = \overline{A}$, i.e., $(A' = A)$. Theorems 4 to 5 determine the conditions for $\overline{A} = \overline{A}$ for two well-known implication functions: Dienes–Rescher and Lukasiewicz.

Theorem 4: The Dienes–Rescher implication satisfies $A' = A = [a_i]_{1 \times n}$ from $B' = B = [b_j]_{1 \times m}$ for the rule “if $x$ is $A$, then $y$ is $B$” when

$$\bigwedge_{i=1}^{n} a_i \geq b_k, \exists k \text{ and } \bigwedge_{j \neq k} \overset{m}{\bigwedge}_{i=1}^{n} b_j \geq \overset{n}{\bigwedge}_{i=1}^{n} a_i,$$

jointly hold.

Proof: The dual of the given rule is “if $y$ is $B$, then $x$ is $\overline{A}$.” Therefore, by (7), we obtain $\overline{A} = BF \circ R$, where $R$ is the relational matrix of the dual rule constructed by the Dienes–Rescher implication function. Replacing $\overline{A} = [\overline{a}_i], BF = [b_j]$ and $R = [b_j \lor \overline{a}_i]$, in (7), we have

$$\overline{a}_i = [b_j] \circ [b_j \lor \overline{a}_i]$$

or

$$\overline{a}_i = \bigvee_{j=1}^{m} (b_j \land (b_j \lor \overline{a}_i))$$

$$= \{b_k \land (b_k \lor \overline{a}_i)\} \lor \bigg\{ \overset{m}{\bigvee}_{j \neq k} \{b_j \land (b_j \lor \overline{a}_i)\} \bigg\}$$

$$= \overline{b}_k \land (b_k \lor \overline{a}_i) = \overline{a}_i,$$

when

$$\overline{b}_k \land (b_k \lor \overline{a}_i) \geq \overline{b}_j \land (b_j \lor \overline{a}_i) \forall j, j \neq k,$$

holds.

Therefore, the conditions stated in Theorem 4 are satisfied, and the theorem should hold. Let us verify it. We first construct the implication relation $R$ using the Dienes–Rescher function and then perform fuzzy modus ponens with $B' = B$ and obtain $A'$, as shown in the following:

$$\overline{A} = BF \circ R$$

$$= [0.8 \ 0.9 \ 0.75] \circ [0.5 \ 0.4 \ 0.3]$$

$$= [0.5 \ 0.4 \ 0.3]$$

$$\therefore A' = [0.5 \ 0.6 \ 0.7] = A.$$

The last result shows that $A'$ has been correctly retrieved. We now derive the condition for correct retrieval by the Lukasiewicz implication function in Theorem 5. We first prove Lemma 1, which is required to prove the theorem.

Lemma 1: Let $R_1$ be a Lukasiewicz-type implication relation for the primal rule “if $x$ is $A$, then $y$ is $B$” and $R_2$ be the same for its dual rule “if $y$ is $B$, then $x$ is $\overline{A}$.” Then, $R_2(y, x) = R_1(x, y)$, for all $x, y$.

Proof: The Lukasiewicz implication relation for the primal rule is given by

$$R_1(x, y) = \text{Min}[1, (1 - \mu_A(x) + \mu_B(y))]$$  \hspace{1cm} (17)

where $\mu_A(x)$ and $\mu_B(y)$ represent the MFs of $x$ is $A$ and $y$ is $B$, respectively. The Lukasiewicz implication function for the dual rule is given by

$$R_2(y, x) = \text{Min}[1, (1 - (1 - \mu_B(y)) + (1 - \mu_A(x)))]$$

$$= \text{Min}[1, (1 - \mu_A(x) + \mu_B(y))]$$  \hspace{1cm} (18)

$$\therefore R_2(y, x) = R_1(x, y)$$ holds for all $x, y$. □

Corollary 1: If $R_1$ and $R_2$ are in matrix form, then the primal and dual implication relations support $R_2 = R_1^T$, where $T$ denotes transposition.

Proof: By Lemma 1, we obtain $R_2(y, x) = R_1(x, y)$. Since it holds for all $x, y$, $R_2 = R_1^T$ follows. □
Theorem 5: The Lukasiewicz implication retrieves \( A' = A = [a_{ij}]_{1 \times n} \) from \( B' = B = [b_{ij}]_{1 \times m} \) for the rule “if \( x \) is \( A \), then \( y \) is \( B \)” if
\[
b_k = 0, \exists k \quad \text{and} \quad \bigwedge_{i=1}^{n} a_i \geq \bigvee_{j=1, j \neq k} b_j
\]
jointly hold.

Proof: Let \( R \) be the Lukasiewicz-type implication relation for the rule “if \( x \) is \( A \), then \( y \) is \( B \).” Then, by Lemma 1, we know that \( R' \) would be the implication relation for the contraposition rule “if \( y \) is \( B \), then \( x \) is \( \overline{A} \).” Thus, by (6)
\[
\overline{A}' = \overline{B}' \circ R'
\]
or \( \overline{a}_i' = \overline{\bigvee}_{j=1}^{m} (b_j \land (1 - a_i + b_j)) \) (by definition of \( r_{ji} \))
\[
= \{b_k \land (1 \land (1 - a_i + b_j))\} \lor \bigvee_{j=1, j \neq k} (b_j \land (1 \land (1 - a_i + b_j)))
\]
= \( 1 - a_i \), if the following conditions jointly hold:

1) \( \overline{b}_k \land (1 \land (1 - a_i + b_j)) \geq \bigvee_{j=1, j \neq k} (b_j \land (1 \land (1 - a_i + b_j))) \)

2) \( b_k = 0 \).

Since \( b_k = 0 \), condition 1 yields
\[
1 - a_i \geq \bigvee_{j=1, j \neq k} (b_j \land (1 \land (1 - a_i + b_j))).
\]

Since \( (1 - a_i) \leq (1 - a_i + b_j) \forall j \), in order to satisfy (20), we need
\[
\overline{b}_j < 1 - a_i + b_j \forall j \neq k.
\]

Substitution of \( \overline{b}_j < 1 - a_i + b_j \) into (20) yields
\[
1 - a_i \geq \bigvee_{j=1, j \neq k} b_j.
\]

Since this has to hold for all \( i \), we write
\[
\bigwedge_{i=1}^{n} a_i \geq \bigvee_{j=1, j \neq k} b_j.
\]

Therefore, \( \overline{a}_i' = 1 - a_i \), if \( b_k = 0, \exists k \) and \( \bigwedge_{i=1}^{n} a_i \geq \bigvee_{j=1, j \neq k} b_j \) hold together.

Definition 1: For any two vectors \( B = [b_{ij}]_{1 \times m} \) and \( B' = [b'_{ij}]_{1 \times m} \), we define \( B' \leq B \), if \( b'_{ij} \leq b_{ij} \forall j \).

Theorem 6: Consider a rule “if \( x \) is \( A \), then \( y \) is \( B \),” where \( A = [a_{ij}]_{1 \times n} \) and \( B = [b_{ij}]_{1 \times m} \). Let the observed MF of the consequent be \( B = [b'_{ij}]_{1 \times m} \) and the inferred MF for abduction be \( A' = [a'_{ij}]_{1 \times n} \). Then, the necessary condition for \( A' = A \) by

Diens–Rescher-type implication-based reasoning with the contraposition rule “if \( y \) is \( B \), then \( x \) is \( \overline{A} \)” is given by \( B \leq \overline{B} \) and
\[
\bigwedge_{i=1}^{n} a_i \geq \bigvee_{j=1}^{m} b'_j.
\]

Proof: Proceeding like the proof of Theorem 5, we arrive at
\[
\overline{a}_i' = \bigvee_{j=1}^{m} (b_j \lor (b_j \lor \overline{a}_j))
\]
\[
\therefore \overline{a}_i' = \overline{a}_j
\]
\[
\text{if} \ (\overline{a}_i \lor b_j) \leq b'_j \ \forall j = 1 \text{ to } m \ \ (21)
\]
and
\[
\overline{a}_i \geq b_j \ \forall j = 1 \text{ to } m. \ \ (22)
\]

Substituting (22) into (21), we have
\[
\overline{a}_i \leq b'_j \ \forall j = 1 \text{ to } m. \ \ (23)
\]

From (22) and (23), we obtain
\[
b_j \leq \overline{a}_i \leq b'_j \ \forall j. \ \ (24)
\]

Since \( b_j \leq b'_j \ \forall j \), we obtain
\[
B \leq \overline{B}. \ \ (25)
\]

Since (23) holds for all \( j \), we have
\[
\overline{a}_i \leq b'_j \ \forall j. \ \ (26)
\]

Again, as (26) holds for all \( i \), we obtain
\[
\bigvee_{i=1}^{n} \overline{a}_i \leq \bigvee_{j=1}^{m} b'_j. \ \ (27)
\]

Taking complement on both sides and simplifying by De Morgan’s law, we find
\[
\bigwedge_{i=1}^{n} a_i \geq \bigvee_{j=1}^{m} b'_j
\]
i.e., given \( B' \), we can obtain \( A' = A \) if \( B \leq \overline{B} \) and \( \bigwedge_{i=1}^{n} a_i \geq \bigvee_{j=1}^{m} b'_j \) jointly hold.

Theorem 7: Given \( B' = [b'_{ij}]_{1 \times m} \) and \( C' = [c'_{ij}]_{1 \times m} \) be two MFs, such that \( b'_i = c'_i + \delta c_i, \delta c_i \geq 0, \forall i = 1, \ldots, m \). Let \( A'_1 \) and \( A'_2 \) be the inferred MFs corresponding to measured MFs for \( B' \) and \( C' \), respectively, obtained by extended contraposition rule: if \( y \) is \( B \), then \( x \) is \( \overline{A} \). If \( B' \geq C' \), then \( A'_1 \geq A'_2 \).

Proof: Since \( A'_1 \) is the inferred MF for the given MF \( B' \), by (6), we have
\[
\overline{A'_1} = \overline{B'} \circ R \ \ (27)
\]
where \( R \) is the relational matrix for the given fuzzy contraposition rule.

Let \( \Delta c = [\delta c_1 \delta c_2 \ldots \delta c_m] \) where \( \delta c_1 \geq 0 \ \forall i \). Now substituting \( B' = C' + \Delta C \) in (27), we have
\[
\overline{A'_1} = C' + \Delta C \circ R
\]
\[
= [(c'_1 + \delta c'_1)(c'_2 + \delta c'_2)\ldots(c'_m + \delta c'_m)] \circ R
\]
\[
= [(1 - (c'_1 + \delta c'_1))\ldots(1 - (c'_m + \delta c'_m))] \circ \{r_{ij}\}_{m \times n}
\]

Since $a_i \geq \frac{n}{j=1} b_j' \forall i$, we have $\bigwedge_{i=1}^{n} a_i \geq \frac{n}{j=1} b_j'$.

Thus, $a_i' \leq a_i$ if $\bigwedge_{i=1}^{n} a_i \geq \frac{n}{j=1} b_j'$.

The theorem follows from (32) and (35).

1) Fuzzy Abduction by Generalized Modus Tollens: The analysis of fuzzy abduction that we have done so far deals with the extended contraposition property. However, abduction can also be performed using GMT. Consider the rule “if $x$ is $A$, then $y$ is $B$.” Suppose we are given the MF of $y$ is $B'$ and that we want to obtain the MF of $x$ is $A'$. Now, by GMT, we have

$$A' = B' \circ R^T$$

where $A' = [a_i']_{1 \times m}, B' = [b_j']_{1 \times m}$ are the MFs of $x$ is $A'$ and $y$ is $B'$, respectively, and $R = [\overline{\pi} \lor b_j]_{n \times m}$ is the implication relation for the given rule by Diens–Rescher implication with $A = [a_i]_{1 \times n}$ and $B = [b_j]_{1 \times m}$. Substituting $A' = [a_i'] , B' = [b_j']$, and $R = [\overline{\pi} \lor b_j]_{n \times m}$ into (36), we have

$$[a_i'] = [b_j'] \circ [b_j \lor \overline{\pi}].$$

Theorem 9 below determines the necessary conditions for abductive retrieval by GMT.

**Theorem 9:** The necessary conditions for $A' = \overline{\pi}$ by GMT for the rule “if $x$ is $A$, then $y$ is $B$” using the Diens–Rescher implication function are $B \leq B'$ and $\bigwedge_{j=1}^{n} \overline{\pi} \geq \bigwedge_{j=1}^{n} b_j$, where $B'$ is the observed MF.

**Proof:** Expanding max-min composition in (37) in pointwise notation, we obtain

$$a_i' = \bigvee_{j} (b_j' \lor \overline{\pi}) = \bigvee_{j} \{b_j' \lor \overline{\pi} \}$$

if the following conditions jointly hold:

$$(b_j' \lor \overline{\pi}) \geq (b_j' \lor \overline{\pi}) \forall j$$

and $\overline{\pi} \leq b_j' \forall j$.

From (38), we obtain

$$\overline{\pi} \geq b_j.$$
Theorem 10: The GMT-based abductive reasoning with the rule “if \( x \) is \( A \), then \( y \) is \( B \)" using Diens–Rescher implication returns

\[
A' \leq \overline{A}, \quad \text{if} \quad \bigwedge_{i} \overline{\alpha_{i}} \geq \bigvee_{j} b_{j}
\]

and

\[
A' \geq \overline{A} \quad \text{if} \quad \bigvee_{i} \overline{\alpha_{i}} \leq \bigvee_{j} \overline{b_{j}} \quad \text{and} \quad \bigwedge_{i} \overline{b_{i}} \geq \bigvee_{j} \overline{\alpha_{i}}
\]

where \( A = [a_{i}] \) and \( B = [b_{j}] \) are measured MFs, and \( A' = [a'_{i}] \) and \( A' = [a'_{j}] \) are observed and inferred MFs.

Proof: Proceedings like the proof of Theorem 9, we obtain

\[
ad'_{i} = \bigvee_{j} (b_{j}' \wedge \overline{b_{j}}) \geq (b_{j}' \wedge \overline{\alpha_{i}})
\]

\[
= \bigvee_{j} (b_{j}' \wedge \overline{\alpha_{i}})
\]

\[
\text{if} \quad (b_{j}' \wedge \overline{\alpha_{i}}) \geq (b_{j}' \wedge b_{j})
\]

\[
or \overline{\alpha_{i}} \geq b_{j}
\]

(43)

or

\[
ad'_{i} = (\bigvee_{j} b_{j}' \wedge \overline{\alpha_{i}})
\]

(44)

Again from (43), we obtain

\[
ad'_{i} = \bigvee_{j} (b_{j}' \wedge \overline{b_{j}})
\]

\[
\text{if} \quad \overline{\alpha_{i}} \leq b_{j}
\]

\[
\geq \bigvee_{j} (b_{j}' \wedge \overline{\alpha_{i}}) \quad \text{or} \quad \overline{\alpha_{i}} \geq \bigvee_{j} b_{j}
\]

(45)

Then

\[
\bigvee_{i} \overline{\alpha_{i}} \geq \bigvee_{j} \overline{b_{j}}
\]

(46)

Therefore, \( a'_{i} \geq \overline{\alpha_{i}} \), if \( \overline{\alpha_{i}} \leq b_{j} \) and \( \overline{\alpha_{i}} \leq \bigvee_{j} b_{j} \) hold jointly \( \forall i, j \)

or \( a'_{i} \geq \overline{\alpha_{i}} \) if

\[
\bigvee_{i} \overline{\alpha_{i}} \leq \bigvee_{j} b_{j} \quad \text{and} \quad \bigvee_{i} \overline{\alpha_{i}} \leq \bigvee_{j} b_{j}'.
\]

B. Rules With Multiple Fuzzy Propositions in Consequent

Here, we determine the conditions for abductive retrieval with rules having multiple fuzzy propositions in consequent for Lukasiewicz- and Diens–Rescher-based implication functions as indicated by Theorems 11 and 12, respectively.

Theorem 11: Let \( A = [a_{i}]_{1 \times n}, B = [b_{j}]_{1 \times m} \) and \( C = [c_{k}]_{1 \times p} \). The contraposition-based fuzzy abduction with the primal rule “if \( x \) is \( A \), then \( y \) is \( B \) or \( z \) is \( C \)" employing Lukasiewicz-type implication function retrieves \( A' \) = \( A \) if the following conditions jointly hold:

1) \( b_{u} = c_{v} = 0 \), for \( \forall u, v \).
2) \( \overline{\alpha_{i}} \geq \bigvee_{j} b_{j} \quad \forall j, k, j \neq u, k \neq v \).

Proof: See the Appendix.

Theorem 12: The contraposition-based fuzzy abduction with the primal rule “if \( x \) is \( A \), then \( y \) is \( B \) or \( z \) is \( C \)" employing the Diens–Rescher-type implication function retrieves \( A' \) = \( A \) if, the following conditions jointly hold:

1) \( b_{u} = c_{v} = 0 \), for \( \forall u, v \).
2) \( \overline{b_{j}} \leq \overline{\alpha_{i}} \quad \forall j, k, j \neq u, k \neq v \).

Proof: See the Appendix.

IV. ABDUCTION WITH MULTIPLE CHAINED RULES

Chaining is an important issue in fuzzy reasoning. Consider two rules \( R_{i} \) and \( R_{j} \), where the consequent part of rule \( R_{i} \) includes \( y_{1} \) is \( B_{1} \) and the antecedent part of rule \( R_{j} \) includes \( y_{1} \) is \( B_{1} \), where \( y_{1} \) is a fuzzy linguistic variable, and \( B_{1} \) and \( B'_{1} \) are two fuzzy sets on the same universe. We call these two rules to be chained (interdependent). The above definition of chaining is explicitly visualized below with the following two rules:

Rule \( R_{i} \): If \( x_{1} \) is \( A_{1} \) and \( x_{2} \) is \( A_{2} \)
then \( y_{1} \) is \( B_{1} \) or \( y_{2} \) is \( B_{2} \)

Rule \( R_{j} \): If \( y_{1} \) is \( B'_{1} \) and \( z_{1} \) is \( C_{1} \)
then \( w_{1} \) is \( D_{1} \) or \( w_{2} \) is \( D_{2} \)

where \( x_{1}, x_{2}, y_{1}, y_{2}, z_{1}, w_{1}, \) and \( w_{2} \) are linguistic variables, and \( A_{1}, A_{2}, B_{1}, B_{2}, B'_{1}, C_{1}, D_{1}, \) and \( D_{2} \) are fuzzy sets (linguistic values) on respective universes.

Petri like nets [57] are probably one of the efficient tools to represent chaining. There exists an extensive literature [58]–[60], [65] on fuzzy reasoning using Petri nets. Researchers prefer Petri like nets in reasoning particularly for their 1) structural advantage in representing knowledge [54] and 2) power of automated reasoning, by which they can automatically identify the “enabled” rules for firing. Here, we would develop a new functional property of Petri nets utilizing its structural properties [61] to design an algorithm for contraposition-based fuzzy abduction. Because of the introduction of the new functional property, the proposed model of computation has operational difference with classical Petri net models [63], and therefore, it is referred to as FLN. The proposed algorithm for fuzzy abduction realized with FLN can perform reasoning even in presence of “complex chaining of rules,” where firing of a third rule depends on the results of joint firing of two (or more) rules. Such complexity in abduction can be handled by imposition of stringent control on rule firing and resource sharing among rules (for example, inferences/resources generated by a rule may be used for firing of other rules). The proposed abductive reasoning
algorithm takes care of all the necessary control actions required to perform abduction with FLN by utilizing its structural and functional properties, and thereby relieves the users from the additional burden of determining the firing sequence of rules and resource sharing.

Here, we propose a primal and dual model of FLN, where the primal net is built up with typical fuzzy IF-THEN rules, which are also called primal rules, whereas the dual net is constructed with a set of rules, obtained by transformation of the rules used in the primal net. The transformation in the present context is fuzzy extension of the contraposition property. After we transform a primal net to its dual form, abduction can be performed on the dual net by the reasoning mechanism introduced earlier (in Section II). The FLN model we shall use for abduction with classical fuzzy sets is described next. An FLN can be defined as a six-tuple:

\[
\text{FLN} = \{P, Tr, D, I, O, R\}
\]

where we have the following.

1. \(P = \{p_1, p_2, \ldots, p_N\}\) is a finite set of places.
2. \(Tr = \{tr_1, tr_2, \ldots, tr_M\}\) is a finite set of transitions.
3. \(D = \{d_1, d_2, \ldots, d_N\}\) is a finite set of fuzzy propositions, where each \(d_i\) is associated with a place \(p_i\).
4. \(I : Tr \rightarrow P\) is the input function, representing a mapping from transitions to their input places.
5. \(O : Tr \rightarrow P\) is the output function, representing a mapping from transitions to their output places.
6. \(n_i : d_i \rightarrow [0, 1]\) is an association function, representing a mapping from fuzzy proposition \(d_i = x_i\) to an \(l\)-dimensional MF \(\mu_{A_i}(x_i)\) describing memberships in \([0,1]\) for \(l\) distinct values of \(x_i\), while \(n_i\) denotes that measured MF for \(x_i\) is \(A_i\), and \(n_i'\) denotes observed MF for the position: \(x_i\) is \(A_i'\).
7. \(R_{ij} : tr_i \times p_j \rightarrow [0, 1] \times [0, 1]\) represents a fuzzy relational matrix \(R_{ij}\) at an arc connected between transition \(tr_i\) to place \(p_j\), where \(p_j \in O(tr_i)\).
8. \(C \subseteq P\) is a set of axioms, where for any \(p_i \in C, p_i \notin O(tr_j), \exists j, tr_j \in Tr\).
9. \(T \subseteq P\) is a set of terminal places, where for any \(p_i \in T, p_i \notin I(tr_j), \exists j, tr_j \in Tr\).

**Example 4**: In Fig. 1, \(P = \{p_1, p_2, \ldots, p_7\}, Tr = \{tr_1, tr_2\}\), and \(D = \{d_1, d_2, \ldots, d_7\}\), where \(d_1 = x_1\) is \(A_1\), \(d_2 = x_2\) is \(A_2\), \(d_3 = y_1\) is \(B_1\), \(d_4 = z_1 = C_1\), \(d_5 = y_2\) is \(B_2\), \(d_6 = w_1\) is \(D_1\), and \(d_7 = w_2\) is \(D_2\).

**Theorem 13**: If \(U\) is a primal FLN constructed with a set of primal rules \(R = \{R_1, R_2, \ldots, R_n\}\), then the dual FLN \(V\) can be constructed by complementing the fuzzy propositions \(d_i\) (i.e., MF of \(d_i\)) at place \(p_i\) for all \(i\) and reversing all the arrows in the primal FLN.

**Proof**: We are given a primal FLN \(U\) containing encoded rules \(R_1, R_2, \ldots, R_n\). Let \(R_i\) be the \(i\)th rule of the form “if \(d_1\) and \(d_2\) and \(\ldots\) and \(d_m\) then \(d_{k+1}\) or \(d_{k+2}\) or \(\ldots\) or \(d_{k+m}\)” encoded in FLN \(U\) using places \(p_1, p_2, \ldots, p_k; p_{k+1}, p_{k+2}, \ldots, p_{k+m}\) and a transition \(tr_i\) such that \(p_j \in I(tr_i) \forall j = 1, 2, \ldots, k\) and \(p_j \in O(tr_i) \forall l = k + 1, k + 2, \ldots, k + m\). When \(R_i\) is transformed into its dual form using the fuzzy contraposition property, we call it \(R_i'\), where \(R_i'\) is given by “if not \(d_{k+1}\) and not \(d_{k+2}\) and \(\ldots\) and not \(d_{k+m}\), then not \(d_1\) or not \(d_2\) or \(\ldots\) or not \(d_m\)”. If \(R_i'\) is encoded into place-transition representation, then \(p_j \in I(tr_i')\), where \(l = k + 1, k + 2, \ldots, k + m\) and \(p_j \in O(tr_i'), j = 1, 2, \ldots, k\), where \(d_i\) is not associated with \(p_i\) and \(d_i\) is not associated with \(p_j\). In other words, \(tr_i'\) and its input and output places can be constructed from \(tr_i\) and its input and output places by reversing arrows and complementing all propositions. Thus, the theorem holds good for independent rules of the primal net.
where \( tr_i \) and \( tr_j \) are the transitions corresponding to rules \( R_i \) and \( R_j \), respectively. After transformation of rules \( R_i \) and \( R_j \) by the fuzzy contraposition property, we obtain new transitions \( tr_i' \) and \( tr_j' \) corresponding to \( tr_i \) and \( tr_j \), respectively, such that \( p_s \in O(tr_i') \) and \( p_s \in I(tr_j') \), which means reversal of arrows in the dual net \( V \) with respect to the primal net around \( p_s \).

Further, because of transformation of the rules \( tr_i \) and \( tr_j \) using the fuzzy contraposition property, the propositions in the input/output places of \( tr_i' \) and \( tr_j' \) would be all complemented with respect to those in the primal net around \( tr_i \) and \( tr_j \). Thus, the proposed theorem is satisfied for chained rules as well.

Now, if we order the rules in set \( R \) in a manner that two successive rules of the set are chained (dependent), then application of the fuzzy contraposition property to these rules results in chained rules with reverse-ordered and negated propositions. The reverse-ordered chained rules thus obtained have a corresponding FLN structure, called the dual net, which can be obtained by reversing arrows of the transitions and negating all propositions in the primal net \( U \). Hence, the theorem follows.

\[ \square \]

### A. Enabling and Firing Condition of Transitions

A transition \( tr_i \) is enabled, if for all \( j, p_j \in I(tr_i) \) possess fuzzy MFs \( n_j \). An enabled transition fires and new MFs are generated at the arc \( tr_i \times p_s \), where \( p_s \in O(tr_i) \).

Let \( R_{i,s} \) be the relational matrix associated with the arc: \( tr_i \times p_s \). Then, the fuzzy truth token (FTT) at the arc \( tr_i \times p_s \) is obtained as \( (\mathcal{T}(n'_j))^\top \circ R_{i,s} \), where \( n'_j \) is the observed MF of the fuzzy proposition \( d_j \) at place \( p_j \). If \( p_s \in k \cap O(tr_i) \) and all \( tr_i \) for \( i = 1 \) to \( k \), for any integer \( k > 1 \), are fired, then the fuzzy MF at place \( p_s \) is obtained as \( n'_s = \bigvee_{i=1}^n \{(\mathcal{T}(n'_j))^\top \circ R_{i,s}\} \), where \( \bigvee \) denotes componentwise ORing of the results \( (\mathcal{T}(n'_j))^\top \circ R_{i,s} \) for each \( i \).

The relational matrix \( R_{i,s} \) evaluated using the Diëns–Rescher implication function is given by \( \{(\mathcal{T}(n_j))^\top \circ \Phi(n_i)\}_j \), where \( \Phi \) denotes Min-Max composition operator, which is computed in the same manner as for Max-Min composition, by swapping the Min and Max operators.

For example, in Fig. 2, we have \( R_{1,s} = (t(n_1, n_2))^\top \Phi(n_2), R_{2,s} = (t(n_3))^\top \Phi(n_3) \) and \( n'_2 = (t(n'_1, n'_2))^\top \circ R_{1,s} \) or \( R_{2,s} \).

It may be noted that like classical Petri nets [61], tokens (here MFs) are not removed from the input places of a fired transition. The principles of enabling and firing of transition, as introduced above for the primal net, are also applicable for the dual net. An algorithm for abduction is presented here based on the enabling and firing conditions of transitions in a dual net.

The proposed algorithm requires the measured MFs at all places of the primal net and observed MFs at the terminal places \( p_i \in \exists j \), where \( p_i \notin I(tr_j) \) \( \exists j \) (i.e., places which are not input places of any transition). The algorithm transforms a given primal net to its dual form by reversing arrows and complementing measured MFs at all places to construct relational matrices associated with the arc: \( tr_j \times p_s \in \forall j \), where \( p_s \in O(tr_j) \). The terminal places of the primal net now becomes axioms \( p_i \in C \).

where \( p_s \notin O(tr_k) \) for the dual net, and observed MFs at the terminal places of the primal net are now complemented. This is the initialization part of the algorithm.

The rest of the algorithm checks the enabling condition of transitions in the dual net and computes FTT at \( tr_j \times p_s \), where \( p_s \in O(tr_j) \), and \( \widehat{\pi}_s \) at place \( p_s \). If \( p_s \) is an output place of two or more transitions \( tr_s(\forall j) \), then until FTT is computed, we cannot compute \( \widehat{\pi}_s \). The algorithm is terminated when the computed MFs at the terminal places of the dual net are obtained. We complement the resulting \( \widehat{\pi}_s \) at the terminal place \( p_i \) in the dual net and thus obtain \( \pi_s \).

### B. Pseudocode of the Fuzzy Abduction Algorithm

1) Transform a primal FLN into its dual form by reversing arrows and complementing measured MF \( n_j \) at each place \( p_i \) of the primal net. In addition, complement the observed MF \( n'_j \) at each terminal place \( p_i \) of the primal FLN and map it at place \( p_i \) of the dual net. Initialize set of current places \( C = \text{set of axioms in the dual net} \) (i.e., set of terminal places in the primal net) and \( T = \text{set of terminal places in the dual net} \) (i.e., set of axioms in the primal net).

2) For each place \( p_i \in C \)

   a) If \( p_i \in I(tr_j) \) \( \exists j \)

   Then, if each input place \( p_k \) of \( tr_j \) possesses MF, then evaluate FTT at the arc \( tr_j \times p_s \), where \( p_s \in O(tr_j) \), as \( \text{FTT}_{j,s} = (\mathcal{T}(n'_k))^\top \circ R_{j,s} \) and \( R_{j,s} = (\mathcal{T}(n_k))^\top \Phi \pi_s \), where \( \Phi \) is Min-Max composition operator.

   End if;

   b) If \( p_s \in O(tr_j) \) and \( \text{FTT}_{j,s} \forall l \) are known, then \( \pi_s \leftarrow \forall l \text{FTT}_{j,s} \).

   If \( p_s \) is the single output place of \( tr_j \), then \( \pi_s \leftarrow \text{FTT}_{j,s} \). If computation of \( \pi_s \) is performed, then \( C \leftarrow C \cup \{p_i\} \);
where 
\[ C = R \circ p \in n \subseteq C(p) \circ \Phi p = \{ p \} \text{ is a companion of } C \]

\[ O \in n \subseteq FTT \circ I \subseteq \Phi \text{ is over } \]

\[ FTT \circ I \subseteq \Phi \text{ is over where } p = R \circ p \subseteq p \]

\[ \text{Computation of } FTT \circ I \subseteq \Phi \text{ is known, where } p = R \circ p \subseteq p \]

\[ \text{Since all input places of } C \text{ are known then } \]

\[ \text{Therefore, we evaluate } \]

\[ \text{Since computation of } \Phi \text{ is over, } C = C \cup \{ p \}. \]

\[ r \subseteq p \subseteq \text{If } \]

\[ \text{For all } C \subseteq \Phi \text{ are known then } \]

\[ \text{Complement the obtained MF of the fuzzy propositions present in set } T. \]

\[ \text{End For;} \]

\[ \text{Repeat step 2 until } C = T. \]

\[ \text{Complement the obtained MF of the fuzzy propositions present in set } T. \]

\[ \text{End.} \]

**C. Trace of the Algorithm**

1) On transformation of the primal net (see Fig. 3), we obtain the dual net (see Fig. 4). We complement \( n_i \) at each place \( p_i \) and complement \( n_i' \) at the axioms of the dual net. Initialize \( C = \{ p_6, p_7, p_5 \} \), and \( T = \{ p_1, p_2 \} \).

2) a) Let \( p_i = p_6 \in I(tr_2) \). All input places \( \{ p_6, p_7 \} \) of \( tr_2 \) possess MFs. Therefore, we evaluate

\[ FTT_{2,3} = t(n_6, n_7')^T \circ R_{2,3} \]

\[ \text{where } R_{2,3} = (t(n_6, n_7')^T \Phi n_3 \]

\[ FTT_{2,4} = t(n_6, n_7')^T \circ R_{2,4} \]

\[ \text{where } R_{2,4} = (t(n_6, n_7')^T \Phi n_4 \]

b) \( p_3 \in O(tr_2) \) only; so, \( \overline{\pi} = FTT_{2,3} \); \( p_4 \in O(tr_2) \cap O(tr_3) \); Therefore, \( \overline{\pi} = FTT_{2,4} \cup FTT_{3,4} \). Since \( FTT_{3,4} \) is not known, the computation of \( \overline{\pi} \) is pending.

Since computation of \( \overline{\pi} \) is over, \( C = C \cup \{ p_3 \}. \)

\[ \text{Similarly as } p_r \text{ is a companion of } p_6, \text{i.e., } \]

\[ p_6, p_7 \in I(tr_2) \text{ only, that is, } \]

\[ p_6, p_7 \notin I(tr_r), j \neq 2; \text{ then, } \]

\[ C \leftarrow C \setminus \{ p_7 \} = \{ p_3, p_5 \}. \]

3) \( C = \{ p_3, p_5 \} \neq \{ p_1, p_2 \}. \) Therefore, repeat from step 2.

2a) \( p_i = p_5 \in I(tr_3) \). Since all input places of \( tr_3 \) possess MFs,

\[ FTT_{3,4} = n_5' \circ R_{3,4} \]

\[ \text{where } R_{3,4} = (n_5')^T \Phi n_4. \]

b) Since \( p_4 \in O(tr_2) \cap O(tr_3), \]

\[ \arrowvert \overline{\pi} = FTT_{2,4} \cup FTT_{3,4} \]

\[ C = C \cup \{ p_4 \} = \{ p_3, p_4, p_5 \}. \]

c) \( p_i = p_6 \in I(tr_3) \) only and \( FTT_{3,4} \) is known, where \( p_4 \in O(tr_3), \]

\[ \arrowvert \overline{\pi} = FTT_{3,4} \]

\[ C \leftarrow C \setminus \{ p_6 \} = \{ p_3, p_1 \}. \]

3) \( C = \{ p_3, p_1 \} \neq T = \{ p_1, p_2 \}. \) Therefore, repeat from step 2.

2a) \( p_i = p_3 \in I(tr_1) \). Since all input places \( \{ p_3, p_4 \} \) of \( tr_1 \) possess MFs, we evaluate

\[ FTT_{1,1} = t(n_3, n_4')^T \circ R_{1,1} \]

\[ \text{where } R_{1,1} = (t(n_3, n_4')^T \Phi n_1 \]

\[ FTT_{1,2} = t(n_3, n_4')^T \circ R_{1,2} \]

\[ \text{where } R_{1,2} = (t(n_3, n_4')^T \Phi n_2. \]

b) \( p_1 \in O(tr_1) \) only, so

\[ \overline{\pi} = FTT_{1,1}; \text{ } p_2 \in O(tr_1) \]

only, so \( \overline{\pi} = FTT_{1,2}; \)

since computation of \( \overline{\pi} \) is over,

\[ C = C \cup \{ p_3, p_2 \} = \{ p_3, p_4, p_1, p_2 \}. \]

c) \( p_i = p_3 \in I(tr_1) \) only, and computation of \( FTT_{1,1} \)

and \( FTT_{1,2} \) is over where \( p_1 \in O(tr_1) \) and \( p_2 \in O(tr_1), \).

\[ \text{As } p_3 \text{ is a companion of } p_3 \text{ and } p_4, \]

\[ \text{move to step 4.} \]
4) Complement the obtained MFs of the fuzzy propositions in set \( T \) to get \( n'_1 \) and \( n'_2 \).
5) End.

D. Time Complexity

Let us consider \( n \) chained primary rules, each with \( m \) fuzzy propositions in the antecedent. The dual Petri net constructed from the primal net thus would have \( n \) rules each with \( m \) fuzzy propositions in the consequent. Since there are \( n \) rules, we should have \( n \) transitions in both the primal and the dual net. Thus, we have \( (n \times m) \) number of implication relational matrices, and therefore, \((m \times n)\) number of steps of abduction to infer the MF [54]. The overall time complexity of the algorithm thus is \( O(mn) \), presuming a uniform cost for composition operation and \( t\)-norm computation for the antecedent.

V. CONCLUSION

The paper examines a new approach to automatic extraction of MF in fuzzy abduction by two-step extension of propositional contraposition property. The first extension lies in the generalization of propositional contraposition property by considering multiple propositions in antecedent and consequent of a rule, while the second extension adds the notion of fuzziness to the contraposition property. The paper offers two fundamental advantages over the existing literature on abduction. The existing GMT-based formulation of fuzzy abduction provides a single relation using \( t\)-norms on the MFs of the fuzzy propositions present in the antecedent of a rule, and thus, the MF of individual fuzzy propositions of the antecedent cannot be separately distinguished/obtained. This has been overcome in this paper with the help of fuzzy extension of the contraposition property. Alternative formulation of fuzzy abduction (designed with a view to providing a better solution) requires evaluation of inverse implication relations. The extensive computations required in evaluating fuzzy inverse relations [11] (with respect to max-min composition operator) to obtain abductive inferences can be avoided by the proposed approach.

Special emphasis is given here to retrieval of MFs due to abduction. Conditions for abductive retrieval for simple rules with single fuzzy proposition in both antecedent and consequent and complex rules with multiple propositions in the consequent have been derived. Theorem 7 provides an interesting observation, which holds for any implication function that supports the fuzzy contraposition property. It states that for a given fuzzy rule “if \( x \) is \( A \), then \( y \) is \( B \),” if \( B' \) is the observed MF and \( A'_1 \) is the inferred MF, then if \( B'_1 \geq B'_2 \), we obtain \( A'_1 \geq A'_2 \).

Finally, this paper demonstrates the scope of abduction through multiple chained rules with the help of an additional structure similar to that of a Petri net. The algorithm used for abduction first transforms a primal network of chained rules to its dual form and then performs reasoning using classical generalized modus ponens. The resulting MFs of the inferred fuzzy propositions are complemented to get back the desired results. A complexity analysis reveals that for \( n \) rules each with \( m \) fuzzy propositions, we need to evaluate \((m \times n)\) number of relational matrices. Consequently, time complexity of the abductive reasoning algorithm is \( O(mn) \), presuming a uniform cost for composition operation and \( t\)-norm computation for the antecedent.

APPENDIX

**Theorem A1:** Relational matrices \( R \) and \( R' \) for \( i = 1 \) to \( n \) constructed by the Lukasiewicz implication function satisfy

\[
R = \sum_{i=1}^{m} R_i(X_1, X_2, \ldots, X_n; Y_i).
\]

**Proof:** With standard definitions of \( \mu_{A_i}(x_i) \) and \( \mu_{B_j}(y_j) \), we construct \( R \) by Lukasiewicz implication and thus obtain

\[
R(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) = \min_{i=1}^{m} \left[ \sum_{j=1}^{m} \mu_{A_i}(x_i) + \mu_{B_j}(y_j) \right]
\]

\[
= \sum_{j=1}^{m} R_i(x_1, x_2, \ldots, x_n; y_j).
\]

Since the last result holds \( \forall x_i \in X_i, i = 1 \) to \( n \), we obtain

\[
R = \sum_{i=1}^{m} S R_i(X_1, X_2, \ldots, X_n; Y_i).
\]

**Theorem A2:** For \( x_i \in X_i \), \( \forall i, y_1 \in Y_1, y_2 \in Y_2 \) and a given

\[
T'(A'_i)
\]

holds when \( R_1 \) and \( R_2 \) are constructed by the Lukasiewicz implication function.

**Proof:** By the Lukasiewicz implication function, we have

\[
R_1(x_1, x_2, \ldots, x_n; y_1) = \min \{1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n)) + \mu_{B_1}(y_1) \}
\]

\[
R_2(x_1, x_2, \ldots, x_n; y_2) = \min \{1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n)) + \mu_{B_2}(y_2) \}
\]

\[
R_1(x_1, x_2, \ldots, x_n; y_1) = \min \{1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n)) + \mu_{B_1}(y_1) s(\mu_{B_2}(y_2)) \}
\]

\[
R_2(x_1, x_2, \ldots, x_n; y_2) = \min \{1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n)) + \mu_{B_1}(y_1) s(\mu_{B_2}(y_2)) \}
\]

Now, let \( T'(A'_i) = \{a'_1, a'_2, \ldots\} \).

Therefore

\[
\bigcap_{i=1}^{n} T'(A'_i) \circ (R_1 \circ R_2)
\]

\[
= \bigcap_{i=1}^{n} [a'_1, a'_2, \ldots] \circ [\min \{1, 1 - (\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n)) + \mu_{B_1}(y_1) s(\mu_{B_2}(y_2)) \}]
\]

\[
= \forall y_1 \in Y_1, y_2 \in Y_2
\]

\[
\bigcap_{i=1}^{n} \left[ \mu_{A_i}(x_i) \bigwedge [\mu_{A_1}(x_1) t \mu_{A_2}(x_2) \ldots t \mu_{A_n}(x_n)) + \mu_{B_1}(y_1) s(\mu_{B_2}(y_2)) \} \right]
\]

\[
= \forall y_1 \in Y_1, y_2 \in Y_2
\]
if the following conditions jointly hold:
\[
(\overline{b}_u \land \overline{c}_v) \land [1 \land \{1 - (\overline{b}_u \land \overline{c}_v) + \overline{\pi}_j\}] \\
\geq \bigvee_{j,k \neq u, k \neq v} (\overline{b}_j \land \overline{c}_k) \land [1 \land \{1 - (\overline{b}_j \land \overline{c}_k) + \overline{\pi}_j\}] 
\]  \hspace{1cm} (A2)

Substituting (A3) into (A2), we have
\[
\overline{\pi}_i \geq \bigvee_{j,k \neq u, k \neq v} \{[\overline{b}_j \land \overline{c}_k] \land [1 \land \{1 - (\overline{b}_j \land \overline{c}_k) + \overline{\pi}_j\}] \}
\]  \hspace{1cm} (A4)

which holds if
\[
\overline{b}_j \land \overline{c}_k \geq \{1 \land \{1 - (\overline{b}_j \land \overline{c}_k) + \overline{\pi}_j\}\}
\]  \hspace{1cm} (A5)

From (A3) and (A7), we thus have the condition of abductive retrieval, rewritten as
1) \(b_u = c_v = 0\),
2) \(\pi_i \geq \overline{b}_j \land \overline{c}_k \forall j, k \neq u, k \neq v.\) \hfill \Box

**Proof of Theorem 12**

Consider the rule “if \(y \rightarrow B\) and \(z \rightarrow C\), then \(x \rightarrow A\).” We use the Lukasiewicz implication rule to construct the relational rule \(R\). Let \(\overline{A} = [\overline{\pi}_i]_{1 \times n}, \overline{B} = [\overline{b}_j]_{1 \times m}, \overline{C} = [\overline{c}_k]_{1 \times p}\), and \(R = [r_{jk,i}]\), where
\[
\begin{align*}
\text{\(r_{jk,i} = \text{Min}\{1, 1 - (\overline{b}_j \land \overline{c}_k) + \overline{\pi}_i\}\) \quad \forall j, k, i.}
\end{align*}
\]

We infer \(A' = [\overline{a}_i]_{1 \times n}, \overline{a}_i = \bigvee_{j,k} (\overline{b}_j \land \overline{c}_k) \land \{1 - (\overline{b}_j \land \overline{c}_k) + \overline{\pi}_j\}\),
\[
\begin{align*}
\overline{b}_j = \overline{b}_j, \overline{c}_k = \overline{c}_k \quad \text{is required for abductive retrieval}
\end{align*}
\]
\[
\begin{align*}
\{\overline{a}_i = \bigvee_{j,k} (\overline{b}_j \land \overline{c}_k) \land [1 \land \{1 - (\overline{b}_j \land \overline{c}_k) + \overline{\pi}_j]\] \\
\forall j, k, l \neq u, k \neq v
\end{align*}
\]

(as for abductive retrieval, we set \(\overline{b}_j = \overline{b}_j\) and \(\overline{c}_k = \overline{c}_k\) )
\[
\begin{align*}
\{\overline{a}_i = \bigvee_{j,k} (\overline{b}_j \land \overline{c}_k) \land [1 \land \{1 - (\overline{b}_j \land \overline{c}_k) + \overline{\pi}_j]\] \\
\forall j, k, l \neq u, k \neq v
\end{align*}
\]

\hspace{1cm} (A8)
If the following conditions hold:
\[
(b_u \wedge c_v) \wedge \left( (b_u \wedge c_v) \vee \nu \right)
\]
\[
\geq \left( (b_j \wedge c_k) \wedge \left( (b_j \wedge c_k) \vee \nu \right) \right), \forall j, k \neq u, k \neq v
\]
\[b_u = 0, \quad c_v = 0. \]  
\[\text{(A9)}\]
Substituting (A10) into (A9), we have
\[
\overline{\alpha} \geq \left( (b_j \wedge c_k) \wedge \left( (b_j \wedge c_k) \vee \nu \right) \right), \forall j, k \neq u, k \neq v
\]
which holds if
\[
\left( b_j \wedge c_k \right) \leq \nu.
\]
\[\text{(A11)}\]
Therefore, from (A10) and (A12), we have the conditions for abductive retrieval, given by
1) \( b_u = 0, c_v = 0 \) for any \( u, v \).
2) \( (b_j \wedge c_k) \leq \nu, \forall j, k, \) except \( j = u, k = v \). □

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